Degenerations of Kählerian K3 surfaces with finite symplectic automorphism groups, II*

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Abstract

We prove the main Conjecture 4 of our paper [18]. Further, we apply these results to classification of degenerations of codimension one of Kählerian K3 surfaces with finite symplectic automorphism groups.

Dedicated to V.P. Platonov on the occasion of his 75th Birthday

1 Introduction

We prove the main Conjecture 4 of our paper [18]. See Theorem 1 below. Further, we apply results of [18], and Theorem 1 and its proof to classification of degenerations of codimension one of Kählerian K3 surfaces with finite symplectic automorphism groups. By classification, we understand an enumeration of connected components of the corresponding moduli. See Theorems 4, 7, 8 below.

Our papers [11] — [18], ideas by K. Hashimoto in [5], results by R. Miranda, D.R. Morrison [8], [9], D.G. James [6] and other results are important to us.

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2 Types of degenerations of codimension 1 of Kählerian K3 surfaces with finite symplectic automorphism groups, and the main classification Theorem 1 which was conjectured in [18, Conjecture 4]

Let X be a Kählerian K3 surface (e. g. see [20], [19], [2], [21], [22] about such surfaces). That is X is a non-singular compact complex surface with the trivial canonical class K_X, and its irregularity $q(X)$ is equal to 0. Then $H^2(X,\mathbb{Z})$ with the intersection pairing is an even unimodular lattice L_{K3} of the signature $(3,19)$. The primitive sublattice $S_X = H^2(X,\mathbb{Z}) \cap H^{1,1}(X) \subset H^2(X,\mathbb{Z})$ is the *Picard lattice* of X generated by first Chern classes of all line bundles over X.

Let G be a finite symplectic automorphism group of X. Here symplectic means that for any $g \in G$, for a non-zero holomoprhic 2-form $\omega_X \in H^{2,0}(X) = \Omega^2[X] = \mathbb{C}\omega_X$, one has $g^*(\omega_X) = \omega_X$. For an G-invariant sublattice $M \subset H^2(X,\mathbb{Z})$, we denote by $M^G = \{x \in K \mid G(x) = x\}$ the *invariant sublattice of* M, and by $M_G = (M^G)^\perp_M$ the *coinvariant sublattice of* M. By [12], the coinvariant lattice $S_G = H^2(X,\mathbb{Z})_G = (S_X)_G$ is *Leech type lattice:* i. e. it is negative definite, it has no elements with square (-2), G acts trivially on the discriminant group A_{S_G}, and $(S_G)^G = \{0\}$. For a general pair (X,G), the $S_G = S_X$, and non-general (X,G) can be considered as Kählerian K3 surfaces with the condition $S_G \subset S_X$ on the Picard lattice (in terminology of [12]). The dimension of their moduli is equal to $20 - \operatorname{rk} S_G$.

Let $E \subset X$ be a non-singular irreducible rational curve (that is $E \cong \mathbb{P}^1$). It is equivalent to: $\alpha = cl(E) \in S_X$, $\alpha^2 = -2$, α is effective and α is numerically effective: $\alpha \cdot D \geq 0$ for every irreducible curve D on X such that $cl(D) \neq \alpha$.

Let us consider the primitive sublattice $S = [S_G, G(\alpha)]_{pr} \subset S_X$ of S_X generated by the coinvariant sublattice S_G and all classes of the orbit $G(E)$. We remind that primitive means that S_X/S has not torsion. Since S_G has no elements with square (-2), it follows that $\operatorname{rk} S = \operatorname{rk} S_G + 1$ and $S = [S_G, \alpha]_{pr} \subset S_X$.

Let us *assume that the lattice $S = [S_G, \alpha]_{pr}$ is negative definite.* Then the elements $G(\alpha)$ define the basis of the root system $\Delta(S)$ of all elements with square (-2) of S. All curves $G(E)$ of X can be contracted to Du Val singularities of types of connected components of the Dynkin diagram of the basis. The group G will act on the corresponding singular K3 surface $\overline{X}$ with these Du Val singularities. For a general such triplet $(X, G, G(E))$, the Picard lattice $S_X = S$, and such triplets can be considered as *a degeneration of codimension* 1 of Kählerian K3 surfaces (X,G) with the finite symplectic automorphism group G. Really, the dimension of moduli of Kählerian K3 surfaces with the condition $S \subset S_X$ on the Picard lattice is equal

to $20 - \operatorname{rk} S = 20 - \operatorname{rk} S_G - 1$.

By Global Torelli Theorem for K3 surfaces [19], [2], the main invariants of the degeneration is the *type of the abstract group* G which is equivalent to the isomorphism class of the coninvariant lattice S_G, and the type of the degeneration which is equivalent to the Dynkin diagram of the basis $G(\alpha)$ or the Dynkin diagram of the rational curves $G(E)$.

We can consider only the maximal finite symplectic automorphism group G with the same coinvariant lattice S_G, that is $G = Clos(G)$. By Global Torelly Theorem for K3 surfaces, this is equivalent to

$$G|S_G = \{g \in O(S_G) \mid g \text{ is identity on } A_{S_G} = (S_G)^*/S_G\}.$$

Indeed, G and $Clos(G)$ have the same lattice S_G, the same orbits $G(E)$ and $Clos(G)(E)$, and the same sublattice $S \subset S_X$.

In [18], all types of $G = Clos(G)$ and types of degenerations (that is Dynkin diagrams of the orbits $G(E)$) are described. They are described in Table 1 below where n gives types of possible $G = Clos(G)$, and we show all possible types of degenerations at the corresponding rows.

Theorem 3 in [18] shows that for a fixed type (defined by $\mathbf{n}$) of abstract finite symplectic group of automorphisms of Kählerian K3 surfaces and for a fixed type of degeneration of codimension 1 (type of Dynkin diagram), the discriminant group $A_S = S^*/S$ of the corresponding lattice S is always the same. It is natural to suppose that the more strong statement is valid: that the isomorphism class of the lattice S is defined uniquely. But, exact calculations and considerations show that it is valid with some few exceptions only (which are given in the Theorem 1 below.

Theorem 1. *Let X be a Kählerian K3 surface with $S_X < 0$. Let $G = Aut^0 X$ be the group of symplectic automorphisms of X (it is always finite) and $P(X)$ be a the set of classes of non-singular rational curves on X. Let $S_G = ((S_X)^G)^\perp_{S_X} \subset S_X$ be the coinvariant sublattice. Assume that S_X is generated by S_G and $P(X)$ up to finite index, and $\operatorname{rk} S_X = \operatorname{rk} S_G + 1$. Then the isomorphism class of the lattice $S = S_X$ is defined uniquely by the type of G as an abstract group (equivalent to $\mathbf{n}$) and the type of the Dynkin diagram of $P(X)$. All possible types of G (equivalent to the invariant $\mathbf{n}$) and Dynkin diagrams $P(X)$ are given in Table 1 below.*

But, it is valid with the following and only the following two exceptions:

(1) $\mathbf{n=34}$ *(equivalently, $G \cong \mathfrak{S}_4$) and the degeneration of the type $6\mathbb{A}_1$;*

(2) $\mathbf{n=10}$ *(equivalently, $H \cong D_8$) and the degeneration of the type $2\mathbb{A}_1$.*

In both these cases there are exactly two isomorphism classes of the lattices S. They (and their genuses) are given in Table 1 below as degenerations $(6\mathbb{A}_1)_I$, $(6\mathbb{A}_1)_{II}$ for $\mathbf{n=34}$*, and $(2\mathbb{A}_1)_I$, $(2\mathbb{A}_1)_{II}$ for* $\mathbf{n=10}$.

This result was conjectured in [18, Conjecture 4]. Below, we give a proof.

3 Discriminant forms technique according to [14]

We use notations, definitions and results of [14] about lattices (that is non-degenerate integral symmetric bilinear forms). Below, we remind the main definitions and results of [14] which we shall use in this paper.

Let S be a lattice, that is a free $\mathbb{Z}$-module of a finite rank with non-degenerate symmetric bilinear form $x\cdot y\in\mathbb{Z}$ for $x,y\in S$. We denote $x^2=x\cdot x$ for $x\in S$. A lattice S is even if x^2 is even for any $x\in S$. For $0\neq\lambda\in\mathbb{Q}$, we denote by $S(\lambda)$ the lattice with symmetric bilinear form $\lambda\, x\cdot y$ for $x,y\in S$ if it is integral.

By $\oplus$, we denote the orthogonal sum of lattices, quadratic forms. For $k\in\mathbb{Z}$ and $k\ge 0$, we denote by kS the orthogonal sum of k copies of a lattice S (in [14], we denoted the same lattice as S^k). We use similar notations for finite quadratic, symmetric bilinear forms.

For a prime p, we denote by $\mathbb{Z}_p$ the ring of p-adic integers, and by $\mathbb{Q}_p$ the field of p-adic numbers.

Let S be a lattice. Then we have the canonical embedding $S\subset S^*=Hom(S,\mathbb{Z})$. It defines the (finite) discriminant group $A_S=S^*/S$ of S. By continuing the symmetric bilinear form of S to S^*, we obtain the non-degenerate finite symmetric bilinear form b_S on A_S with values in $\mathbb{Q}/\mathbb{Z}$ and the finite quadratic form q_S on A_S with values in $\mathbb{Q}/2\mathbb{Z}$ if S is even. They are called *discriminant forms of S.*

We denote by $l(A)$ the minimal number of generators of a finite Abelian group A, and by $|A|$ its order. For a prime p, we denote by $q_{S_p}=q_{S\otimes\mathbb{Z}_p}$ the p-component of q_S (equivalently, the discriminant quadratic form of the p-adic lattice $S\otimes\mathbb{Z}_p$).

A p-adic lattice $K(q_{S_p})$ or the rank $l(A_{S_p})$ with the discriminant quadratic form q_{S_p} is denoted by $K(q_{S_p})$. It is unique, up to isomorphisms, for $p\neq 2$, and for $p=2$, if $q_{S_2}\not\cong q_\theta^{(2)}(2)\oplus q'$ where $q_\theta^{(2)}(2)$ denotes a finite quadratic form on a group of order 2 (see notations below). For $p=2$ and $q_{S_2}\cong q_\theta^{(2)}(2)\oplus q'$, there are exactly two lattices $K(q_{S_2})$; their determinants are different by 5 mod $(\mathbb{Z}_2^*)^2$. See [14, Theorem 1.9.1].

By $< A >$ we denote a lattice defined by a matrix A. In particular, $K_\theta^{(p)}(p^k)=\langle\theta p^k\rangle$, $\theta\in\mathbb{Z}_p^*$, and $U^{(2)}(2^k)=\left\langle\begin{array}{cc}0 & 2^k\\ 2^k & 0\end{array}\right\rangle$, $V^{(2)}(2^k)=\left\langle\begin{array}{cc}2^{k+1} & 2^k\\ 2^k & 2^{k+1}\end{array}\right\rangle$ are standard p-adic lattices of the rank 1 and 2. By Jordan decomposition (e.g. see [3]), any p-adic lattice is their orthogonal sum. By $q_\theta^{(p)}(p^k)$ and $b_\theta^{(p)}(p^k)$ we denote discriminant quadratic and bilinear forms of $K_\theta^{(p)}(p^k)$ respectively. By $u_+(2^k)$, $u_-(2^k)$ and $v_+(2^k)$, $v_-(2^k)$ we denote discriminant quadratic, bilinear forms of $U^{(2)}(2^k)$ and $V^{(2)}(2^k)$ respectively.

One has relations between these forms from [14, Prop. 1.8.2]:

$$Relations\ (a) - (k): \tag{1}$$

(a) $2K_\theta^{(p)}(p^k) \cong 2K_{\theta'}^{(p)}(p^k)$ if $p \neq 2$;
(b) $2U^{(2)}(2^k) \cong 2V^{(2)}(2^k)$;
(c) $K_\theta^{(2)}(2^k) \oplus K_{\theta'}^{(2)}(2^k) \cong K_{5\theta}^{(2)}(2^k) \oplus K_{5\theta'}^{(2)}(2^k)$;
(d) $2K_\theta^{(2)}(2^k) \oplus K_{\theta'}^{(2)}(2^k) \cong \begin{cases} V^{(2)}(2^k) \oplus K_{-5\theta'}^{(2)}(2^k) & if\ \theta' \equiv \theta(\bmod 4), \\ U^{(2)}(2^k) \oplus K_{-\theta'}^{(2)}(2^k) & if\ \theta' \equiv -\theta(\bmod 4) \end{cases}$;
(e) $V^{(2)}(2^k) \oplus K_\theta^{(2)}(2^{k+1}) \cong U^{(2)}(2^k) \oplus K_{5\theta}^{(2)}(2^{k+1})$;
(f) $K_\theta^{(2)}(2^k) \oplus V^{(2)}(2^{k+1}) \cong K_{5\theta}^{(2)}(2^k) \oplus U^{(2)}(2^{k+1})$;
(g) $K_\theta^{(2)}(2^k) \oplus K_{\theta'}^{(2)}(2^{k+1}) \cong K_{\theta+2\theta'}^{(2)}(2^k) \oplus K_{5(\theta'-2\theta)}^{(2)}(2^{k+1})$;
(h) $K_\theta^{(2)}(2^k) \oplus K_{\theta'}^{(2)}(2^{k+2}) \cong K_{5\theta}^{(2)}(2^k) \oplus K_{5\theta'}^{(2)}(2^{k+2})$;
(i) $q_\theta^{(2)}(2) \cong q_{5\theta}^{(2)}(2)$;
(j) $b_\theta^{(2)}(2) \cong b_{\theta'}^{(2)}(2)$, $u_-^{(2)}(2) \cong v_-^{(2)}(2)$, $b_\theta^{(2)}(4) \cong b_{5\theta}^{(2)}(4)$;
(k) relations between finite quadratic and bilinear forms which follow from (a) — (h) if one considers discriminant forms.

By [14], the numbers $(t_{(+)}, t_{(-)})$ of positive, negative squares of $S\otimes\mathbb{R}$ and the discriminant quadratic form q_S define the *genus* of an even lattice S, that is isomorphism classes of $S\otimes\mathbb{R}$ and $S\otimes\mathbb{Z}_p$ for all prime p.

If S is an even lattice, the signature $sign\ S = t_{(+)} - t_{(-)}$ of S modulo 8 that is $t_{(+)} - t_{(-)} \bmod 8 \equiv sign\ q_S \bmod 8$ is the invariant of q_S. In particular, $sign\ S \equiv 0 \bmod 8$ if S is unimodular and even. We have (e. g. see [14, Prop. 1.11.2])

$$sign\ q_\theta^{(p)}(p^k) \equiv k^2(1-p) + 4k\eta \bmod 8,\ where\ p \neq 2,\ and\ (-1)^\eta = \left(\frac{\theta}{p}\right); \tag{2}$$

$$sign\ q_\theta^{(2)}(2^k) \equiv \theta + 4\omega(\theta)k \bmod 8,\ where\ \omega(\theta) \equiv \frac{\theta^2-1}{8} \bmod 2; \tag{3}$$

$$sign\ u_+^{(2)}(2^k) \equiv 0 \bmod 8,\quad sign\ v_+^{(2)}(2^k) \equiv 4k \bmod 8. \tag{4}$$

In particular,

$$sign\ q_\theta^{(2)}(4) \equiv \theta \bmod 8,\quad sign\ u_+^{(2)}(4) \equiv sign\ v_+^{(2)}(4) \equiv 0 \bmod 8. \tag{5}$$

To find the genus of a lattice S, we consider the Jordan decomposition

$$S^{(p)} = S\otimes\mathbb{Z}_p = \bigoplus_{k\geq 0} S_k^{(p)}(p^k) \tag{6}$$

where $S_k^{(p)}$ are unimodular p-adic lattices. Then the p-component q_{S_p} is orthogonal sum of discriminant quadratic forms of $S_k^{(p)}(p^k)$ for $k \geq 1$ and non-zero $S_k^{(p)}$.

For $p \neq 2$, by relations (a) — (k) above,

$$S_k^{(p)} = (m_k - 1)K_1^{(p)}(1) \oplus K_{\theta_k}^{(p)}(1) \tag{7}$$

where $m_k = \operatorname{rk} S_k^{(p)}$ (or m_k is the size of $S_k^{(p)}$), and $\theta_k = \det(S_k^{(p)}) \in \mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$. Thus, $S_k^{(p)}$ is defined by invariants $m_k = \operatorname{rk} S_k^{(p)}$ and the Kronecker symbol $\left(\frac{\theta_k}{p}\right)$ where $\theta_k = \det S_k^{(p)}$. Then $q_{S_k^{(p)}(p^k)} = (m_k - 1)q_1^{(p)}(p^k) \oplus q_{\theta_k}^{(p)}(p^k)$. Like in [3], we shortly denote this form by the symbol $(p^k)^{\pm m_k}$ where $\pm 1 = \left(\frac{\theta_k}{p}\right)$.

For $p = 2$, the lattice $S_k^{(2)}$ has the type II if it is orthogonal sum of lattices $U^{(2)}(1)$ and $V^{(2)}(1)$, and it has the type I otherwise. Up to isomorphisms, $S_k^{(2)}$ is defined by the type, size=rk $S_k^{(2)}$, det = det $S_k^{(2)} \in \mathbb{Z}_2^*/(\pm)(\mathbb{Z}_2^*)^2 = \{1, 5\} \bmod (\pm)(\mathbb{Z}_2^*)^2$, and *sign* mod 8 $\equiv$ *sign* $q_{S_k^{(2)}(4)}$ mod 8. See (5) about calculations of *sign* mod 8. Shortly, like in [3], $S_k^{(2)}(2^k)$ is denoted by $(2^k)_{II}^{(-1)^\delta \cdot s}$ for $S_k^{(2)}$ of the type II, size=s, det $= 5^\delta \bmod (\pm)(\mathbb{Z}_2^*)^2$, and by $(2^k)_{sign \bmod 8}^{(-1)^\delta \cdot s}$ for $S_k^{(2)}$ of type I, size=s, det $= 5^\delta \bmod (pm)(\mathbb{Z}_2^*)^2$ and with *sign* mod 8. For the Jordan decomposition (6), these components are separated by comme , instead of $\oplus$. The same notations are used for the corresponding discriminant quadratic forms.

In Sect. 8, we give Program 8 which uses these invariants to calculate the genus of a lattice given by an integral symmetric matrix l.

The following fact from [14] will be very important for us.

Proposition 2. *(Proposition 1.6.1 from [14]) Primitive embeddings of an even lattice M into unimodular even lattices such that the orthogonal complement to M is isomorphic to K are defined by isomorphisms $\gamma : A_M \cong A_K$ such that $q_K \circ \gamma = -q_M$. Two such isomorphims γ, γ' define isomorphic primitive embeddings if and only if they are conjugate by an automorphism of K, and they define isomorphic primitive sublattices if and only if $\gamma \circ \overline{\phi} = \overline{\psi} \circ \gamma'$ for some $\phi \in O(M)$, $\psi \in O(K)$.*

4 A proof of Theorem 1

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We remind that Niemeier lattices are negative definite even unimodular lattices of the rank 24. There are 24 such lattices, up to isomorphisms, $N = N_j$, $j = 1, 2, \ldots, 24$, classified by Niemeier. They are characterized by their root sublattices $N^{(2)}$ generated by all their

elements with square (-2) (called roots). Further, $\Delta(N)$ is the set of all roots of N. We have the following list of Niemeier lattices N_j where the number j is shown in the bracket:

$$N^{(2)} = [\Delta(N)] =$$

$$(1)\ D_{24},\ (2)\ D_{16}\oplus E_8,\ (3)\ 3E_8,\ (4)\ A_{24},\ (5)\ 2D_{12},\ (6)\ A_{17}\oplus E_7,\ (7)\ D_{10}\oplus 2E_7,$$

$$(8)\ A_{15}\oplus D_9,\ (9)\ 3D_8,\ (10)\ 2A_{12},\ (11)\ A_{11}\oplus D_7\oplus E_6,\ (12)\ 4E_6,\ (13)\ 2A_9\oplus D_6,$$

$$(14)\ 4D_6,\ (15)\ 3A_8,\ (16)\ 2A_7\oplus 2D_5,\ (17)\ 4A_6,\ (18)\ 4A_5\oplus D_4,\ (19)\ 6D_4,$$

$$(20)\ 6A_4,\ (21)\ 8A_3,\ (22)\ 12A_2,\ (23)\ 24A_1$$

give 23 Niemeier lattices N_j. The last is Leech lattice (24) with $N^{(2)} = \{0\}$ which has no roots. Further, $N(R)$ denotes the Niemeier lattice with the root system R. We fix the basis $P(N)$ of the root system $\Delta(N)$ of N. By $A(N) \subset O(N)$ we denote the subgroup of the group $O(N)$ of automorphisms of N which permute the basis $P(N)$.

Let us consider the lattice S for one of types (given by $\mathbf{n}$) of the group G and for a type of the degeneration given by a Dynkin diagram R. All such possibilities are enumerated in Table 1.

We consider all possible markings $S \subset N_j$ by Niemeier lattices. It means that: $S \subset N_j$ is a primitive sublattice that is N_j/S has no torsion; $P(S) \subset P(N_j)$.

By [14, Remark 1.14.7],

$$G = \{g \in A(N_j) \mid g|S^{\perp}_{N_j} = identity\} \subset A(N_j).$$

It follows that we can find all such lattices S as follows.

Find a Niemeier lattice N_j, a subgroup $G \subset A(N_j)$ (up to conjugacy in $A(N_j)$), an element $\alpha \in P(N_j)$ such that $G(\alpha)$ has the Dynkin diagram R, and the primitive sublattice $S = [(N_j)_G, \alpha]_{pr} \subset N_j$ has a primitive embedding into L_{K3}. Then S, $G|S$ and $G(\alpha) \subset P(S)$ correspond to a degeneration of codimenion one of some Kählerian K3 surfaces by Global Torelli Theorem and epimorphicity of Torelli map for Kählerian K3 surfaces [2], [7], [19], [21], [22].

Obviously, the isomorphism class of S does not change if G is changed by conjugacy in $A(N_j)$, the element α is changed to $h(\alpha)$ by $h \in Normalizer(A(N_j), G)$.

All such triplets (N_j, G, α) (up to isomorphisms) are shown in columns of Table 2 below using results of [16], [17] and [18] and the program GAP, [4]. In Table 2, for all possible $\mathbf{n}$ and the type R (Dynkin diagram) of the degeneration, the first line of the column gives j, the second line gives the group $G = H_{n,t} \subset A(N_j)$ (in notations of [16], [17] and [18]). The third line gives $\alpha \in P(N_j)$ which give different orbits $G(\alpha) \subset P(N_j)$ with the Dynkin diagram

of the type R, but these orbits are conjugate by the $Normalizer(A(N_j), G = H_{n,t})$. Thus, $S = [(N_j)_{H_{n,t}}, \alpha]_{pr} \subset N_j$, up to isomorphisms. The fourth line gives the Dynkin diagram of the root system $(S_{N_j}^{\perp})^{(2)}$ of elements with square (-2) in $S_{N_j}^{\perp}$. The fifth line (if it is necessary) gives the number of elements with square (-4) in $S_{N_j}^{\perp}$.

In Table 1 below we calculate the genus of the lattices S for all possible triplets (N_j, G, α) (equivalently, columns of Table 2), using invariants and relations (1) of Sect. 3. We use Programs 7 and 8 from Appendix, Sect. 8. The genus is defined by the types $\mathbf{n}$ and R of the degeneration except two cases. For $\mathbf{n}= 10$, $R = 2\mathbb{A}_1$ and $\mathbf{n}= 34$, $R = 6\mathbb{A}_1$ there are two possibilities for the genus which we label by $(2\mathbb{A}_1)_I$, $(2\mathbb{A}_1)_{II}$ and by $(6\mathbb{A}_1)_I$, $(6\mathbb{A}_1)_{II}$ respectively. Here I and II show the type of the corresponding 2-adic lattices. In Table 1, we also give the genus of the lattice S_G which was calculated by K. Hashimoto in [5] (it is useful to compare genuses of S_G and S).

Let us consider a case $(\mathbf{n}, R)$ which is different from $(4, \mathbb{A}_1)$ and $(16, \mathbb{A}_1)$. In Table 2, one of columns of this case $(\mathbf{n}, R)$ is marked by $*$ from above. We denote the lattice S of this case by $\mathbf{S}$. The orthogonal complement $\mathbf{S}_{N_j}^{\perp}$ of this case either has the root system $(\mathbf{S}_{N_j}^{\perp})^{(2)}$ of elements with square (-2) which is different from all other columns of this case, or it has different number of elements with square (-4) (the last happens for $(10, (2\mathbb{A}_1)_{II})$, $(34, (6\mathbb{A}_1)_{II})$ and $(51, 8\mathbb{A}_1)$). Since genuses of lattices S of all other columns and the genus of $\mathbf{S}$ are the same, by Proposition 2, there exists an isomorphism $\gamma : A_S \cong A_{\mathbf{S}_{N_j}^{\perp}}$ such that $q_S \circ \gamma = -q_{\mathbf{S}_{N_j}^{\perp}}$. By Proposition 2, this defines a primitive embedding $S \subset N$ into one of Niemeier lattices N such that $S_N^{\perp} = \mathbf{S}_{N_j}^{\perp}$. Changing this embedding by $w \in W(N)$ for the reflection group $W(N)$ of N, if necessary, we can assume that $P(S) \subset P(N)$ and $S \subset N$ is isomorphic to one of columns of the case $(\mathbf{n}, R)$. Since $S_N^{\perp} \cong \mathbf{S}_{N_j}^{\perp}$ and the column with this property is unique, we obtain that $S \cong \mathbf{S}$. Thus, the lattice S is unique up to isomorphisms.

For the cases $(\mathbf{n} = 4, \mathbb{A}_1)$ and $(\mathbf{n} = 16, \mathbb{A}_1)$ (and all other cases $(\mathbf{n}, \mathbb{A}_1)$ as well) we have that $K = S_G \oplus \langle -2\rangle \subset S$ is an overlattice of a finite index, by definition. Discriminant groups of S_G and their orders are is known (e.g. see [5]) (they can be found from the Table 1). Discriminant groups of S and their orders are calculated in [18] (they can be found from the Table 1). It follows that orders of the discriminant groups of the lattices K and S are the same. It follows that $S = K = S_G \oplus \langle -2\rangle$, and it is unique up to isomorphisms since the lattices S_G are unique up to isomorphisms by [5].

This finishes the proof of Theorem 1.

Table 1: Genuses of degenerations of codimension 1 of Kählerian K3 surfaces with finite symplectic automorphism groups $G = Clos(G)$.

n	$\|G\|$	i	G	rk S_G	q_{S_G}	Deg	rk S	q_S
1	2	1	C_2	8	2_{II}^{+8}	$\mathbb{A}_1$	9	2_7^{+9}
						$2\mathbb{A}_1$	9	$2_{II}^{-6}, 4_3^{-1}$
2	3	1	C_3	12	3^{+6}	$\mathbb{A}_1$	13	$2_3^{-1}, 3^{+6}$
						$3\mathbb{A}_1$	13	$2_1^{+1}, 3^{-5}$
3	4	2	C_2^2	12	$2_{II}^{-6}, 4_{II}^{-2}$	$\mathbb{A}_1$	13	$2_3^{+7}, 4_{II}^{+2}$
						$2\mathbb{A}_1$	13	$2_{II}^{-4}, 4_7^{-3}$
						$4\mathbb{A}_1$	13	$2_{II}^{-6}, 8_3^{-1}$
4	4	1	C_4	14	$2_2^{+2}, 4_{II}^{+4}$	$\mathbb{A}_1$	15	$2_5^{-3}, 4_{II}^{+4}$
						$2\mathbb{A}_1$	15	4_1^{-5}
						$4\mathbb{A}_1$	15	$2_2^{+2}, 4_{II}^{+2}, 8_7^{+1}$
						$\mathbb{A}_2$	15	$2_1^{+1}, 4_{II}^{-4}$
6	6	1	D_6	14	$2_{II}^{-2}, 3^{+5}$	$\mathbb{A}_1$	15	$2_7^{-3}, 3^{+5}$
						$2\mathbb{A}_1$	15	$4_3^{-1}, 3^{+5}$
						$3\mathbb{A}_1$	15	$2_1^{-3}, 3^{-4}$
						$6\mathbb{A}_1$	15	$4_1^{+1}, 3^{+4}$
9	8	5	C_2^3	14	$2_{II}^{+6}, 4_2^{+2}$	$2\mathbb{A}_1$	15	$2_{II}^{-4}, 4_5^{-3}$
						$4\mathbb{A}_1$	15	$2_{II}^{+6}, 8_1^{+1}$
						$8\mathbb{A}_1$	15	$2_{II}^{+6}, 4_1^{+1}$
10	8	3	D_8	15	4_1^{+5}	$\mathbb{A}_1$	16	$2_1^{+1}, 4_7^{+5}$
						$(2\mathbb{A}_1)_I$	16	$2_6^{-2}, 4_6^{-4}$
						$(2\mathbb{A}_1)_{II}$	16	$2_{II}^{+2}, 4_{II}^{+4}$
						$4\mathbb{A}_1$	16	$4_7^{+3}, 8_1^{+1}$
						$8\mathbb{A}_1$	16	4_0^{+4}
						$2\mathbb{A}_2$	16	4_{II}^{+4}
12	8	4	Q_8	17	$2_7^{-3}, 8_{II}^{-2}$	$8\mathbb{A}_1$	18	$2_7^{-3}, 16_3^{-1}$
						$\mathbb{A}_2$	18	$2_6^{-2}, 8_{II}^{-2}$
16	10	1	D_{10}	16	5^{+4}	$\mathbb{A}_1$	17	$2_7^{+1}, 5^{+4}$
						$5\mathbb{A}_1$	17	$2_7^{+1}, 5^{-3}$

n	$\lvert G\rvert$	i	G	$\mathrm{rk}\, S_G$	q_{S_G}	Deg	$\mathrm{rk}\, S$	q_S
17	12	3	$\mathfrak{A}_4$	16	$2_{II}^{-2}, 4_{II}^{-2}, 3^{+2}$	$\mathbb{A}_1$	17	$2_7^{-3}, 4_{II}^{+2}, 3^{+2}$
						$3\mathbb{A}_1$	17	$2_1^{-3}, 4_{II}^{+2}, 3^{-1}$
						$4\mathbb{A}_1$	17	$2_{II}^{-2}, 8_3^{-1}, 3^{+2}$
						$6\mathbb{A}_1$	17	$4_1^{-3}, 3^{+1}$
						$12\mathbb{A}_1$	17	$2_{II}^{-2}, 8_1^{+1}, 3^{-1}$
18	12	4	D_{12}	16	$2_{II}^{+4}, 3^{+4}$	$\mathbb{A}_1$	17	$2_7^{+5}, 3^{+4}$
						$2\mathbb{A}_1$	17	$2_{II}^{+2}, 4_7^{+1}, 3^{+4}$
						$3\mathbb{A}_1$	17	$2_5^{-5}, 3^{-3}$
						$6\mathbb{A}_1$	17	$2_{II}^{-2}, 4_1^{+1}, 3^{+3}$
21	16	14	C_2^4	15	$2_{II}^{+6}, 8_I^{+1}$	$4\mathbb{A}_1$	16	$2_{II}^{+4}, 4_{II}^{+2}$
						$16\mathbb{A}_1$	16	2_{II}^{+6}
22	16	11	$C_2 \times D_8$	16	$2_{II}^{+2}, 4_0^{+4}$	$2\mathbb{A}_1$	17	4_7^{+5}
						$4\mathbb{A}_1$	17	$2_{II}^{+2}, 4_0^{+2}, 8_7^{+1}$
						$8\mathbb{A}_1$	17	$2_{II}^{+2}, 4_7^{+3}$
26	16	8	SD_{16}	18	$2_7^{+1}, 4_7^{+1}, 8_{II}^{+2}$	$8\mathbb{A}_1$	19	$2_1^{+1}, 4_1^{+1}, 16_3^{-1}$
						$2\mathbb{A}_2$	19	$2_5^{-1}, 8_{II}^{-2}$
30	18	4	$\mathfrak{A}_{3,3}$	16	$3^{+4}, 9^{-1}$	$3\mathbb{A}_1$	17	$2_5^{-1}, 3^{-3}, 9^{-1}$
						$9\mathbb{A}_1$	17	$2_3^{-1}, 3^{+4}$
32	20	3	$Hol(C_5)$	18	$2_6^{-2}, 5^{+3}$	$2\mathbb{A}_1$	19	$4_1^{+1}, 5^{+3}$
						$5\mathbb{A}_1$	19	$2_1^{+3}, 5^{-2}$
						$10\mathbb{A}_1$	19	$4_5^{-1}, 5^{+2}$
						$5\mathbb{A}_2$	19	$2_5^{-1}, 5^{-2}$
33	21	1	$C_7 \rtimes C_3$	18	7^{+3}	$7\mathbb{A}_1$	19	$2_1^{+1}, 7^{+2}$
34	24	12	$\mathfrak{S}_4$	17	$4_3^{+3}, 3^{+2}$	$\mathbb{A}_1$	18	$2_5^{-1}, 4_1^{+3}, 3^{+2}$
						$2\mathbb{A}_1$	18	$2_2^{+2}, 4_{II}^{+2}, 3^{+2}$
						$3\mathbb{A}_1$	18	$2_7^{+1}, 4_5^{-3}, 3^{-1}$
						$4\mathbb{A}_1$	18	$4_3^{-1}, 8_3^{-1}, 3^{+2}$
						$(6\mathbb{A}_1)_I$	18	$2_4^{-2}, 4_0^{+2}, 3^{+1}$
						$(6\mathbb{A}_1)_{II}$	18	$2_{II}^{+2}, 4_{II}^{-2}, 3^{+1}$
						$8\mathbb{A}_1$	18	$4_2^{+2}, 3^{+2}$
						$12\mathbb{A}_1$	18	$4_5^{-1}, 8_7^{+1}, 3^{-1}$
						$6\mathbb{A}_2$	18	$4_{II}^{-2}, 3^{+1}$

n	$\lvert G\rvert$	i	G	$\mathrm{rk}\, S_G$	q_{S_G}	Deg	$\mathrm{rk}\, S$	q_S
39	32	27	2^4C_2	17	$2_{II}^{+2}, 4_0^{+2}, 8_7^{+1}$	$4\mathbb{A}_1$	18	4_6^{+4}
						$8\mathbb{A}_1$	18	$2_{II}^{+2}, 4_7^{+1}, 8_7^{+1}$
						$16\mathbb{A}_1$	18	$2_{II}^{+2}, 4_6^{+2}$
40	32	49	$Q_8 * Q_8$	17	4_7^{+5}	$8\mathbb{A}_1$	18	4_6^{+4}
46	36	9	3^2C_4	18	$2_6^{-2}, 3^{+2}, 9^{-1}$	$6\mathbb{A}_1$	19	$4_7^{+1}, 3^{+1}, 9^{-1}$
						$9\mathbb{A}_1$	19	$2_5^{-3}, 3^{+2}$
						$9\mathbb{A}_2$	19	$2_5^{-1}, 3^{+2}$
48	36	10	$\mathfrak{S}_{3,3}$	18	$2_{II}^{-2}, 3^{+3}, 9^{-1}$	$3\mathbb{A}_1$	19	$2_5^{+3}, 3^{-2}, 9^{-1}$
						$6\mathbb{A}_1$	19	$4_1^{+1}, 3^{+2}, 9^{-1}$
						$9\mathbb{A}_1$	19	$2_7^{-3}, 3^{+3}$
49	48	50	2^4C_3	17	$2_{II}^{-4}, 8_1^{+1}, 3^{-1}$	$4\mathbb{A}_1$	18	$2_{II}^{-2}, 4_{II}^{+2}, 3^{-1}$
						$12\mathbb{A}_1$	18	$2_{II}^{-2}, 4_2^{-2}$
						$16\mathbb{A}_1$	18	$2_{II}^{-4}, 3^{-1}$
51	48	48	$C_2 \times \mathfrak{S}_4$	18	$2_{II}^{+2}, 4_2^{+2}, 3^{+2}$	$2\mathbb{A}_1$	19	$4_1^{+3}, 3^{+2}$
						$4\mathbb{A}_1$	19	$2_{II}^{+2}, 8_1^{+1}, 3^{+2}$
						$6\mathbb{A}_1$	19	$4_7^{-3}, 3^{+1}$
						$8\mathbb{A}_1$	19	$2_{II}^{-2}, 4_5^{-1}, 3^{+2}$
						$12\mathbb{A}_1$	19	$2_{II}^{-2}, 8_7^{+1}, 3^{-1}$
55	60	5	$\mathfrak{A}_5$	18	$2_{II}^{-2}, 3^{+1}, 5^{-2}$	$\mathbb{A}_1$	19	$2_7^{-3}, 3^{+1}, 5^{-2}$
						$5\mathbb{A}_1$	19	$2_3^{+3}, 3^{+1}, 5^{+1}$
						$6\mathbb{A}_1$	19	$4_1^{+1}, 5^{-2}$
						$10\mathbb{A}_1$	19	$4_7^{+1}, 3^{+1}, 5^{-1}$
						$15\mathbb{A}_1$	19	$2_5^{+3}, 5^{-1}$
56	64	138	$\Gamma_{25}a_1$	18	$4_5^{+3}, 8_1^{+1}$	$8\mathbb{A}_1$	19	$4_4^{-2}, 8_5^{-1}$
						$16\mathbb{A}_1$	19	4_5^{+3}
61	72	43	$\mathfrak{A}_{4,3}$	18	$4_{II}^{-2}, 3^{-3}$	$3\mathbb{A}_1$	19	$2_5^{-1}, 4_{II}^{+2}, 3^{+2}$
						$12\mathbb{A}_1$	19	$8_1^{+1}, 3^{+2}$
65	96	227	2^4D_6	18	$2_{II}^{-2}, 4_7^{+1}, 8_1^{+1}, 3^{-1}$	$4\mathbb{A}_1$	19	$4_3^{-3}, 3^{-1}$
						$8\mathbb{A}_1$	19	$2_{II}^{-2}, 8_7^{+1}, 3^{-1}$
						$12\mathbb{A}_1$	19	4_5^{+3}
						$16\mathbb{A}_1$	19	$2_{II}^{+2}, 4_3^{-1}, 3^{-1}$
75	192	1023	$4^2\mathfrak{A}_4$	18	$2_{II}^{-2}, 8_6^{-2}$	$16\mathbb{A}_1$	19	$2_{II}^{-2}, 8_5^{-1}$

Table 2: Lattices $S_{N_j}^{\perp}$ for markings by Niemeier lattices N_j.

n=1, degeneration $\mathbb{A}_1$:

j	3	3	3	3	3	3	3	3	6	6
H	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
orbit of	$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$	$\alpha_{5,3}$	$\alpha_{6,3}$	$\alpha_{7,3}$	$\alpha_{8,3}$	$\alpha_{1,2}$	$\alpha_{2,2}$
$(S_{N_j}^{\perp})^{(2)}$	E_7	E_7	E_7	E_7	E_7	E_7	E_7	E_7	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$

6	6	6	6	6	6	7
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$	$\alpha_{6,2}$	$\alpha_{7,2}$	$\alpha_{9,1}$	$\alpha_{1,1}$
$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$8A_1 \oplus E_7$	$D_7 \oplus A_1$

7	7	7	7	7	7	7	8
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\alpha_{6,1}$	$\alpha_{7,1}$	$\alpha_{8,1}$	$\alpha_{1,2}$
$D_7 \oplus A_1$	$D_7 \oplus A_1$	$D_7 \oplus A_1$	$D_7 \oplus A_1$	$D_7 \oplus A_1$	$D_7 \oplus A_1$	$D_7 \oplus A_1$	$9A_1 \oplus D_6$

8	8	8	8	8	8	8
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$	$\alpha_{6,2}$	$\alpha_{7,2}$	$\alpha_{8,1}$
$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$7A_1 \oplus D_8$

9	9	9	9	9	9	9	9
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$	$\alpha_{5,3}$	$\alpha_{6,3}$	$\alpha_{7,3}$	$\alpha_{8,3}$
$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$	$A_1 \oplus D_6$

11	11	11	11	11	11
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{6,1}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$
$5A_1 \oplus D_4 \oplus D_6$	$7A_1 \oplus 2D_4$	$7A_1 \oplus 2D_4$	$7A_1 \oplus 2D_4$	$7A_1 \oplus 2D_4$	$7A_1 \oplus 2D_4$

11	11	12	12	12	12
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,2}$	$H_{1,2}$
$\alpha_{2,3}$	$\alpha_{4,3}$	$\alpha_{2,1}$,$\alpha_{2,2}$,$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{4,1}$,$\alpha_{4,2}$,$\alpha_{4,3}$,$\alpha_{4,4}$	$\alpha_{2,1}$	$\alpha_{4,1}$
$9A_1 \oplus D_6$	$9A_1 \oplus D_6$	$3A_1 \oplus 3D_4$	$3A_1 \oplus D_4$	$3A_1 \oplus E_6$	$3A_1 \oplus E_6$

12	12	12	12	13	13	13
$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,2}$,$\alpha_{6,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$,$\alpha_{5,2}$	$\alpha_{4,2}$	$\alpha_{5,1}$,$\alpha_{5,2}$	$\alpha_{1,3}$	$\alpha_{2,3}$
$A_5 \oplus D_4$	$A_5 \oplus D_4$	$A_5 \oplus D_4$	$A_5 \oplus D_4$	$9A_1 \oplus D_6$	$11A_1 \oplus D_4$	$11A_1 \oplus D_4$

13	13	13	14	14	14
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{3,3}$	$\alpha_{4,3}$	$\alpha_{5,3}$,$\alpha_{6,3}$	$\alpha_{1,1}$,$\alpha_{1,2}$	$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{3,1}$,$\alpha_{3,2}$
$11A_1 \oplus D_4$	$11A_1 \oplus D_4$	$11A_1 \oplus D_4$	$A_1 \oplus A_3 \oplus D_5$	$A_1 \oplus A_3 \oplus D_5$	$A_1 \oplus A_3 \oplus D_5$

14	15	15	15	15	16	16	16
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{4,1}$,$\alpha_{4,2}$	$\alpha_{1,1}$,$\alpha_{8,1}$	$\alpha_{2,1}$,$\alpha_{7,1}$	$\alpha_{3,1}$,$\alpha_{6,1}$	$\alpha_{4,1}$,$\alpha_{5,1}$	$\alpha_{1,1}$,$\alpha_{7,1}$	$\alpha_{2,1}$,$\alpha_{6,1}$	$\alpha_{3,1}$,$\alpha_{5,1}$
$A_1 \oplus A_3 \oplus D_5$	A_6	A_6	A_6	A_6	$4A_1 \oplus A_5$	$4A_1 \oplus A_5$	$4A_1 \oplus A_5$

16	16	16	16	16	16	16
$H_{1,1}$	$H_{1,1}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,3}$
$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,1}$,$\alpha_{4,2}$	$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{3,3}$,$\alpha_{3,4}$	$\alpha_{1,3}$
$4A_1 \oplus A_5$	$3A_1 \oplus A_7$	$7A_1 \oplus 2D_4$	$11A_1 \oplus D_4$	$11A_1 \oplus D_4$	$11A_1 \oplus D_4$	$A_1 \oplus A_3 \oplus D_4$

16	16	16	16	16	16
$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$
$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$,$\alpha_{5,3}$	$\alpha_{1,4}$	$\alpha_{2,4}$	$\alpha_{3,4}$
$A_1 \oplus A_3 \oplus D_4$	$A_1 \oplus A_3 \oplus D_4$	$A_1 \oplus A_3 \oplus D_4$	$3A_1 \oplus D_5$	$3A_1 \oplus D_5$	$3A_1 \oplus D_5$

18	18	18	18	18	18
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,1}$,$\alpha_{5,1}$	$\alpha_{2,1}$,$\alpha_{4,1}$	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{2,5}$	$\alpha_{4,5}$
$3A_1 \oplus 2A_3$	$3A_1 \oplus 2A_3$	$3A_1 \oplus 2A_3$	$2A_1 \oplus A_3 \oplus A_5$	$4A_1 \oplus A_5$	$4A_1 \oplus A_5$

18	18	18	19	19
$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{3,1}$,$\alpha_{3,2}$,$\alpha_{3,3}$,$\alpha_{3,4}$	$\alpha_{1,5}$,$\alpha_{3,5}$,$\alpha_{4,5}$	$\alpha_{2,5}$	$\alpha_{2,3}$,$\alpha_{2,4}$,$\alpha_{2,5}$,$\alpha_{2,6}$	$\alpha_{4,3}$,$\alpha_{4,4}$,$\alpha_{4,5}$,$\alpha_{4,6}$
$11A_1 \oplus D_4$	$15A_1$	$15A_1$	$A_1 \oplus 3A_3$	$A_1 \oplus 3A_3$

19	19	20	20
$H_{1,2}$	$H_{1,2}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{4,3}$,$\alpha_{1,4}$,$\alpha_{3,4}$,$\alpha_{4,4}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{4,1}$,$\alpha_{1,2}$,$\alpha_{4,2}$	$\alpha_{2,1}$,$\alpha_{3,1}$,$\alpha_{2,2}$,$\alpha_{3,2}$
$3A_1 \oplus D_4$	$3A_1 \oplus D_4$	$A_2 \oplus A_4$	$A_2 \oplus A_4$

21	21	21	21
$H_{1,1}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$
$\alpha_{2,1},\alpha_{2,2},\alpha_{2,3},\alpha_{2,4},\alpha_{2,5},\alpha_{2,6},\alpha_{2,7},\alpha_{2,8}$	$\alpha_{1,1},\alpha_{3,1},\alpha_{1,5},\alpha_{3,5}$	$\alpha_{2,1},\alpha_{2,5}$	$\alpha_{2,2},\alpha_{2,4}$
$15A_1$	$5A_1 \oplus A_3$	$5A_1 \oplus A_3$	$3A_1 \oplus 2A_3$

22	23 $*$
$H_{1,1}$	$H_{1,1}$
$\alpha_{1,1},\alpha_{2,1},\alpha_{1,2},\alpha_{2,2},\alpha_{1,5},\alpha_{2,5},\alpha_{1,11},\alpha_{2,11}$	$\alpha_{1},\alpha_{4},\alpha_{8},\alpha_{12},\alpha_{15},\alpha_{17},\alpha_{18},\alpha_{21}$
$3A_2$	$7A_1$

n=1, degeneration $2\mathbb{A}_1$:

j	3	3	3	3	3	3	3	3	6	6
H	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\alpha_{6,1}$	$\alpha_{7,1}$	$\alpha_{8,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$
$(S_{N_j}^{\perp})^{(2)}$	E_8	E_8	E_8	E_8	E_8	E_8	E_8	E_8	$7A_1 \oplus E_7$	$7A_1 \oplus E_7$

6	6	6	6	6	6	7	7
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\alpha_{6,1}$	$\alpha_{7,1}$	$\alpha_{8,1}$	$\alpha_{9,1}$	$\alpha_{1,2}$
$7A_1 \oplus E_7$	$7A_1 \oplus E_7$	$7A_1 \oplus E_7$	$7A_1 \oplus E_7$	$7A_1 \oplus E_7$	$7A_1 \oplus E_7$	D_8	D_9

7	7	7	7	7	7	8	8	8	8
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$	$\alpha_{6,2}$	$\alpha_{7,2}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$
D_9	D_9	D_9	D_9	D_9	D_9	$6A_1 \oplus D_8$	$6A_1 \oplus D_8$	$6A_1 \oplus D_8$	$6A_1 \oplus D_8$

8	8	8	8	9	9	9	9	9	9
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{5,1}$	$\alpha_{6,1}$	$\alpha_{7,1}$	$\alpha_{8,2}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{5,1}$	$\alpha_{6,1}$
$6A_1 \oplus D_8$	$6A_1 \oplus D_8$	$6A_1 \oplus D_8$	$8A_1 \oplus D_7$	D_8	D_8	D_8	D_8	D_8	D_8

9	9	11	11	11	11
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{7,1}$	$\alpha_{8,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$
D_8	D_8	$4A_1 \oplus D_4 \oplus D_6$	$4A_1 \oplus D_4 \oplus D_6$	$4A_1 \oplus D_4 \oplus D_6$	$4A_1 \oplus D_4 \oplus D_6$

11	11	11	11	12
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{5,1}$	$\alpha_{6,2}$	$\alpha_{1,3}$	$\alpha_{3,3}$	$\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{1,3}$,$\alpha_{1,4}$
$4A_1 \oplus D_4 \oplus D_6$	$6A_1 \oplus D_4 \oplus D_5$	$6A_1 \oplus A_3 \oplus D_6$	$6A_1 \oplus A_3 \oplus D_6$	$A_3 \oplus 3D_4$

12	12	12	12	12	12	12
$H_{1,1}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$
$\alpha_{3,1}$,$\alpha_{3,2}$,$\alpha_{3,3}$,$\alpha_{3,4}$	$\alpha_{1,1}$	$\alpha_{1,3}$,$\alpha_{6,3}$	$\alpha_{3,1}$	$\alpha_{2,3}$	$\alpha_{3,3}$,$\alpha_{5,3}$	$\alpha_{4,3}$
$A_3 \oplus 3D_4$	$A_3 \oplus E_6$	$D_4 \oplus E_6$	$A_3 \oplus D_6$	$D_4 \oplus E_6$	$D_4 \oplus E_6$	$D_4 \oplus E_6$

13	13	13	13	14	14	14	14	14
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,1}$,$\alpha_{1,2}$	$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{3,1}$,$\alpha_{3,2}$	$\alpha_{4,1}$,$\alpha_{4,2}$	$\alpha_{5,1}$,$\alpha_{5,2}$	$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$
$8A_1 \oplus D_6$	$8A_1 \oplus D_6$	$8A_1 \oplus D_6$	$8A_1 \oplus D_6$	$D_4 \oplus D_5$	$2D_5$	$2D_5$	$2D_5$	$2D_5$

14	15	15	15	15	16	16	16
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{5,3}$,$\alpha_{6,3}$	$\alpha_{1,2}$,$\alpha_{8,2}$	$\alpha_{2,2}$,$\alpha_{7,2}$	$\alpha_{3,2}$,$\alpha_{6,2}$	$\alpha_{4,2}$,$\alpha_{5,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$
$2D_5$	A_8	A_8	A_8	A_8	$2A_1 \oplus A_7$	$2A_1 \oplus A_7$	$2A_1 \oplus A_7$

16	16	16	16	16	16	16
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$
$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$,$\alpha_{5,3}$	$\alpha_{1,1}$,$\alpha_{1,2}$	$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{3,1}$,$\alpha_{3,2}$
$4A_1 \oplus A_7$	$4A_1 \oplus A_7$	$4A_1 \oplus A_7$	$4A_1 \oplus A_7$	$6A_1 \oplus 2D_4$	$6A_1 \oplus 2D_4$	$6A_1 \oplus 2D_4$

16	16	16	16	16	16	18
$H_{1,2}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,3}$	$H_{1,1}$
$\alpha_{4,3}$,$\alpha_{4,4}$	$\alpha_{1,1}$,$\alpha_{7,1}$	$\alpha_{2,1}$,$\alpha_{6,1}$	$\alpha_{3,1}$,$\alpha_{5,1}$	$\alpha_{4,1}$	$\alpha_{4,4}$	$\alpha_{1,2}$
$8A_1 \oplus A_3 \oplus D_4$	$D_4 \oplus D_5$	$D_4 \oplus D_5$	$D_4 \oplus D_5$	$D_4 \oplus D_5$	$A_3 \oplus D_5$	$A_1 \oplus A_3 \oplus A_5$

18	18	18	18	18
$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{2,2}$	$\alpha_{1,3}$,$\alpha_{5,3}$	$\alpha_{2,3}$,$\alpha_{4,3}$	$\alpha_{3,3}$	$\alpha_{1,5}$
$A_1 \oplus A_3 \oplus A_5$	$3A_1 \oplus A_3 \oplus A_5$	$3A_1 \oplus A_3 \oplus A_5$	$3A_1 \oplus A_3 \oplus A_5$	$5A_1 \oplus A_5$

18	18	19	19	19
$H_{1,2}$	$H_{1,2}$	$H_{1,1}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{2,1}$,$\alpha_{2,2}$,$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{1,3}$,$\alpha_{1,4}$,$\alpha_{1,5}$,$\alpha_{1,6}$
$10A_1 \oplus D_4$	$10A_1 \oplus D_4$	$4A_3$	$4A_3$	$2A_1 \oplus 3A_3$

19	19	20	20
$H_{1,2}$	$H_{1,2}$	$H_{1,1}$	$H_{1,1}$
$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$,$\alpha_{1,5}$,$\alpha_{3,5}$,$\alpha_{4,5}$	$\alpha_{2,1}$,$\alpha_{2,5}$	$\alpha_{1,3}$,$\alpha_{4,3}$,$\alpha_{1,4}$,$\alpha_{4,4}$	$\alpha_{2,3}$,$\alpha_{3,3}$,$\alpha_{2,4}$,$\alpha_{3,4}$
$2D_4$	$2D_4$	$2A_4$	$2A_4$

21	21	21	21
$H_{1,1}$	$H_{1,2}$	$H_{1,2}$	$H_{1,2}$
$\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{1,3}$,$\alpha_{1,4}$,$\alpha_{1,5}$,$\alpha_{1,6}$,$\alpha_{1,7}$,$\alpha_{1,8}$	$\alpha_{1,2}$,$\alpha_{1,4}$	$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{1,7}$,$\alpha_{3,7}$	$\alpha_{2,3}$,$\alpha_{2,7}$
$14A_1$	$2A_1 \oplus 2A_3$	$4A_1 \oplus 2A_3$	$4A_1 \oplus 2A_3$

22	23 $*$
$H_{1,1}$	$H_{1,1}$
$\alpha_{1,3}$,$\alpha_{2,3}$,$\alpha_{1,6}$,$\alpha_{2,6}$,$\alpha_{1,7}$,$\alpha_{2,7}$,$\alpha_{1,9}$,$\alpha_{2,9}$	α_{2},α_{3},α_{5},α_{6},α_{7},α_{10},α_{11},α_{14}
$4A_2$	$8A_1$

n=2, degeneration $\mathbb{A}_1$:

j	12	12	12	12	14	14	14	14
H	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
orbit of	$\alpha_{1,1}$,$\alpha_{6,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$,$\alpha_{5,1}$	$\alpha_{4,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$
$(S_{N_j}^{\perp})^{(2)}$	A_5	A_5	A_5	A_5	$A_1 \oplus D_4$	$A_1 \oplus D_4$	$A_1 \oplus D_4$	$A_1 \oplus D_4$

14	17	17	17	18	18	18	18
$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
$\alpha_{5,1}$,$\alpha_{6,1}$	$\alpha_{1,1}$,$\alpha_{6,1}$	$\alpha_{2,1}$,$\alpha_{5,1}$	$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{1,1}$,$\alpha_{5,1}$	$\alpha_{2,1}$,$\alpha_{4,1}$	$\alpha_{3,1}$	$\alpha_{2,5}$
$A_1 \oplus D_4$	A_4	A_4	A_4	$A_2 \oplus A_3$	$A_2 \oplus A_3$	$A_2 \oplus A_3$	A_5

19	19	19	19	21
$H_{2,1}$	$H_{2,2}$	$H_{2,2}$	$H_{2,2}$	$H_{2,1}$
$\alpha_{2,1}$,$\alpha_{2,2}$,$\alpha_{2,3}$,$\alpha_{2,4}$,$\alpha_{2,5}$,$\alpha_{2,6}$	$\alpha_{2,3}$,$\alpha_{2,5}$	$\alpha_{1,4}$,$\alpha_{3,4}$,$\alpha_{4,4}$	$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{1,2}$,$\alpha_{3,2}$
$5A_2$	$A_2 \oplus D_4$	$3A_1 \oplus 2A_2$	$3A_1 \oplus 2A_2$	$A_1 \oplus A_3$

21	22	23 $*$
$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{1,6}$,$\alpha_{2,6}$,$\alpha_{1,8}$,$\alpha_{2,8}$,$\alpha_{1,10}$,$\alpha_{2,10}$	α_{3},α_{4},α_{14},α_{17},α_{21},α_{24}
$A_1 \oplus A_3$	$2A_2$	$5A_1$

n=2, degeneration $3\mathbb{A}_1$:

j	12	12	12	12	14	14	14	14	14
H	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
orbit of	$\alpha_{1,2}$,$\alpha_{6,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$,$\alpha_{5,2}$	$\alpha_{4,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$	$\alpha_{4,2}$	$\alpha_{5,2}$,$\alpha_{6,2}$
$(S_{N_j}^{\perp})^{(2)}$	E_6	E_6	E_6	E_6	D_6	D_6	D_6	D_6	D_6

17	17	17	18	18	18	18
$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
$\alpha_{1,2}$,$\alpha_{6,2}$	$\alpha_{2,2}$,$\alpha_{5,2}$	$\alpha_{3,2}$,$\alpha_{4,2}$	$\alpha_{1,2}$,$\alpha_{5,2}$	$\alpha_{2,2}$,$\alpha_{4,2}$	$\alpha_{3,2}$	$\alpha_{1,5}$
A_6	A_6	A_6	$A_2 \oplus A_5$	$A_2 \oplus A_5$	$A_2 \oplus A_5$	$A_1 \oplus A_5$

19	19	19	19	21
$H_{2,1}$	$H_{2,2}$	$H_{2,2}$	$H_{2,2}$	$H_{2,1}$
$\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{1,3}$,$\alpha_{1,4}$,$\alpha_{1,5}$,$\alpha_{1,6}$	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{1,3}$,$\alpha_{1,5}$	$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{1,5}$,$\alpha_{3,5}$
$A_1 \oplus 5A_2$	$2A_2 \oplus D_4$	$2A_2 \oplus D_4$	$A_1 \oplus A_2 \oplus D_4$	$2A_3$

21	22	23 $*$
$H_{2,1}$	$H_{2,1}$	$H_{2,1}$
$\alpha_{2,3}$,$\alpha_{2,5}$	$\alpha_{1,1}$,$\alpha_{2,1}$,$\alpha_{1,2}$,$\alpha_{2,2}$,$\alpha_{1,7}$,$\alpha_{2,7}$	α_1,α_2,α_5,α_6,α_{12},α_{15}
$2A_3$	$3A_2$	$6A_1$

n=3, degeneration $\mathbb{A}_1$:

j	12	12	16	16	16	16	18
H	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,2}$	$H_{3,2}$	$H_{3,2}$	$H_{3,1}$
orbit of	$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{4,1}$,$\alpha_{4,2}$	$\alpha_{4,1}$,$\alpha_{4,2}$	$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{3,3}$,$\alpha_{3,4}$	$\alpha_{3,1}$,$\alpha_{3,2}$
$(S_{N_j}^{\perp})^{(2)}$	$3A_1 \oplus D_4$	$3A_1 \oplus D_4$	$7A_1$	$3A_1 \oplus D_4$	$3A_1 \oplus D_4$	$3A_1 \oplus D_4$	$5A_1 \oplus A_3$

18	18	19	19	19	19	21
$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,2}$	$H_{3,2}$	$H_{3,1}$
$\alpha_{2,5}$	$\alpha_{4,5}$	$\alpha_{2,5}$,$\alpha_{2,6}$	$\alpha_{4,5}$,$\alpha_{4,6}$	$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{4,3}$,$\alpha_{1,4}$,$\alpha_{3,4}$,$\alpha_{4,4}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{2,1}$,$\alpha_{2,2}$,$\alpha_{2,4}$,$\alpha_{2,5}$
$7A_1$	$7A_1$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$3A_1 \oplus D_4$	$3A_1 \oplus D_4$	$7A_1$

21	21	21	21	21	21	22 $*$
$H_{3,2}$	$H_{3,2}$	$H_{3,2}$	$H_{3,3}$	$H_{3,3}$	$H_{3,4}$	$H_{3,1}$
$\alpha_{1,1}$,$\alpha_{3,1}$	$\alpha_{2,1}$	$\alpha_{2,2}$,$\alpha_{2,4}$,$\alpha_{2,5}$	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{1,5}$,$\alpha_{3,5}$	$\alpha_{2,1}$,$\alpha_{2,5}$	$\alpha_{2,1}$,$\alpha_{2,5}$	$\alpha_{1,1}$,$\alpha_{2,1}$,$\alpha_{1,2}$,$\alpha_{2,2}$
$7A_1$	$7A_1$	$5A_1 \oplus A_3$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$3A_1$	A_2

23	23
$H_{3,1}$	$H_{3,2}$
$\alpha_1,\alpha_5,\alpha_6,\alpha_{10},\alpha_{15},\alpha_{17},\ \alpha_{21},\alpha_{24}$	$\alpha_1,\alpha_5,\alpha_6,\alpha_{21}$
$7A_1$	$3A_1$

n=3, degeneration $2\mathbb{A}_1$:

j	12	12	12	12	16	16	16	16	16
H	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$
orbit of	$\alpha_{1,1},\alpha_{1,2}$	$\alpha_{3,1},\alpha_{3,2}$	$\alpha_{2,3}$	$\alpha_{4,3}$	$\alpha_{1,1},\alpha_{1,2}$	$\alpha_{2,1},\alpha_{2,2}$	$\alpha_{3,1},\alpha_{3,2}$	$\alpha_{1,3}$	$\alpha_{2,3}$
$(S_{N_j}^{\perp})^{(2)}$	$A_3\oplus D_4$	$A_3\oplus D_4$	$2D_4$	$2D_4$	$6A_1$	$6A_1$	$6A_1$	$8A_1$	$8A_1$

16	16	16	18	18	18 $*$	18	19	19
$H_{3,1}$	$H_{3,2}$	$H_{3,2}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$
$\alpha_{3,3}$	$\alpha_{4,1}$	$\alpha_{4,3},\alpha_{4,4}$	$\alpha_{1,1},\alpha_{1,2}$	$\alpha_{2,1},\alpha_{2,2}$	$\alpha_{3,3}$	$\alpha_{1,5}$	$\alpha_{2,1},\alpha_{2,3}$	$\alpha_{4,1},\alpha_{4,3}$
$8A_1$	$2D_4$	$A_3\oplus D_4$	$4A_1\oplus A_3$	$4A_1\oplus A_3$	$6A_1\oplus A_3$	$8A_1$	$2A_3$	$2A_3$

19	19	19	21
$H_{3,1}$	$H_{3,3}$	$H_{3,3}$	$H_{3,1}$
$\alpha_{1,5},\alpha_{1,6}$	$\alpha_{1,1},\alpha_{3,1},\alpha_{4,1},\ \alpha_{1,3},\alpha_{3,3},\alpha_{4,3},\alpha_{1,5},\alpha_{3,5},\alpha_{4,5}$	$\alpha_{2,1},\alpha_{2,3},\alpha_{2,5}$	$\alpha_{1,1},\alpha_{1,2},\alpha_{1,4},\alpha_{1,5}$
$2A_1\oplus A_3$	$\{0\}$	$\{0\}$	$6A_1$

21	21	21	21	21	21	21
$H_{3,1}$	$H_{3,2}$	$H_{3,3}$	$H_{3,4}$	$H_{3,4}$	$H_{3,4}$	$H_{3,4}$
$\alpha_{2,3},\alpha_{2,7}$	$\alpha_{1,2},\alpha_{1,4},\alpha_{1,5}$	$\alpha_{2,2},\alpha_{2,3},\alpha_{2,7}$	$\alpha_{1,1},\alpha_{1,5}$	$\alpha_{2,2}$	$\alpha_{1,3},\alpha_{3,3},\ \alpha_{1,7},\alpha_{3,7}$	$\alpha_{2,3},\alpha_{2,7}$
$8A_1$	$4A_1\oplus A_3$	$2A_3$	$2A_1$	$4A_1$	$4A_1$	$4A_1$

22	23	23
$H_{3,1}$	$H_{3,2}$	$H_{3,3}$
$\alpha_{1,3},\alpha_{2,3},\alpha_{1,5},\alpha_{2,5},\alpha_{1,7},\alpha_{2,7}$	$\alpha_8,\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{15},\alpha_{16}$	$\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_{11},\alpha_{12},\alpha_{15},\alpha_{16}$
$2A_2$	$4A_1$	$\{0\}$

n=3, degeneration $4\mathbb{A}_1$:

j	12	12	16	16	16	16	18	18	19	19
H	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,2}$	$H_{3,2}$	$H_{3,2}$	$H_{3,1}$	$H_{3,1}$	$H_{3,1}$	$H_{3,2}$
orbit of	$\alpha_{1,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{1,1},\alpha_{1,3}$	$\alpha_{2,1}$
$(S_{N_j}^{\perp})^{(2)}$	$2D_4$	$2D_4$	$8A_1$	$2D_4$	$2D_4$	$2D_4$	$6A_1\oplus A_3$	$6A_1\oplus A_3$	$2A_3$	$2D_4$

19	21	21	21	21	21	22 $*$
$H_{3,2}$	$H_{3,1}$	$H_{3,2}$	$H_{3,2}$	$H_{3,3}$	$H_{3,4}$	$H_{3,1}$
$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{1,3}$,$\alpha_{1,7}$	$\alpha_{1,3}$,$\alpha_{3,3}$	$\alpha_{2,3}$	$\alpha_{1,2}$, $\alpha_{1,3}$,$\alpha_{1,7}$	$\alpha_{1,2}$	$\alpha_{1,6}$,$\alpha_{2,6}$
$2D_4$	$6A_1$	$6A_1 \oplus A_3$	$6A_1 \oplus A_3$	$2A_3$	$4A_1$	$2A_2$

23	23
$H_{3,1}$	$H_{3,2}$
α_2,α_3,α_4,α_7	α_2,α_4
$8A_1$	$4A_1$

n=4, degeneration $\mathbb{A}_1$:

j	13	13	13	13	18	18	18	19
H	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$
orbit of	$\alpha_{1,3}$	$\alpha_{2,3}$	$\alpha_{3,3}$	$\alpha_{4,3}$	$\alpha_{1,5}$,$\alpha_{3,5}$	$\alpha_{2,5}$	$\alpha_{4,5}$	$\alpha_{2,3}$,$\alpha_{2,4}$
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$3A_1$	$3A_1$	$3A_1$	$A_1 \oplus A_3$

19	20	20	21	21	21	22
$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$
$\alpha_{3,3}$,$\alpha_{1,4}$	$\alpha_{1,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$,$\alpha_{3,1}$	$\alpha_{2,1}$	$\alpha_{1,2}$,$\alpha_{3,2}$	$\alpha_{2,2}$	$\alpha_{1,2}$,$\alpha_{2,2}$,$\alpha_{1,5}$,$\alpha_{2,5}$
$A_1 \oplus A_3$	$2A_1 \oplus A_2$	$2A_1 \oplus A_2$	$A_1 \oplus A_3$	$3A_1$	$3A_1$	$2A_1 \oplus A_2$

23
$H_{4,1}$
α_1,α_4,α_{17},α_{18}
$3A_1$

n=4, degeneration $2\mathbb{A}_1$:

j	13	13	18	19	20	21	21	23 $*$
H	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$
orbit of	$\alpha_{5,1}$	$\alpha_{5,3}$	$\alpha_{3,1}$,$\alpha_{3,3}$	$\alpha_{1,3}$,$\alpha_{3,4}$	$\alpha_{1,2}$	$\alpha_{1,1}$	$\alpha_{2,3}$	α_8,α_{12}
$(S_{N_j}^{\perp})^{(2)}$	D_5	D_4	D_4	$2A_1 \oplus A_3$	A_4	A_3	$2A_1 \oplus A_3$	$4A_1$

n=4, degeneration $4\mathbb{A}_1$:

j	13	13	13	13	18	18	19	19	19	20
H	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{4,1}$	$\alpha_{1,1}$,$\alpha_{1,3}$	$\alpha_{2,1}$,$\alpha_{2,3}$	$\alpha_{1,1}$,$\alpha_{3,1}$	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{1,3}$,$\alpha_{4,3}$
$(S_{N_j}^{\perp})^{(2)}$	D_5	D_5	D_5	D_5	D_4	D_4	$2A_3$	$2A_3$	$2A_3$	$2A_1 \oplus A_4$

20	21	21	21	22	23 $*$
$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$	$H_{4,1}$
$\alpha_{2,3}$,$\alpha_{3,3}$	$\alpha_{1,3}$	$\alpha_{1,4}$,$\alpha_{3,4}$	$\alpha_{2,4}$	$\alpha_{1,3}$,$\alpha_{2,3}$,$\alpha_{1,6}$,$\alpha_{2,6}$	α_2,α_3,α_7,α_{11}
$2A_1 \oplus A_4$	$2A_1 \oplus A_3$	$2A_1 \oplus A_3$	$2A_1 \oplus A_3$	$2A_1 \oplus 2A_2$	$4A_1$

n=4, degeneration $\mathbb{A}_2$:

j	20	22 $*$
H	$H_{4,1}$	$H_{4,1}$
orbit of	$\alpha_{2,2}$	$\alpha_{1,1}$,$\alpha_{1,11}$
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_4$	$A_1 \oplus 2A_2$

n=6, degeneration $\mathbb{A}_1$:

j	12	12	12	12	12	12	18	18	18
H	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,2}$	$H_{6,2}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$
orbit of	$\alpha_{1,1}$,$\alpha_{6,1}$	$\alpha_{2,1}$	$\alpha_{3,1}$,$\alpha_{5,1}$	$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{1,1}$,$\alpha_{5,1}$	$\alpha_{2,1}$,$\alpha_{4,1}$	$\alpha_{3,1}$
$(S_{N_j}^{\perp})^{(2)}$	A_5	A_5	A_5	A_5	$3A_1$	$3A_1$	$A_2 \oplus A_3$	$A_2 \oplus A_3$	$A_2 \oplus A_3$

18	18	18	19	19	19	19
$H_{6,1}$	$H_{6,2}$	$H_{6,2}$	$H_{6,1}$	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$
$\alpha_{2,5}$	$\alpha_{3,1}$	$\alpha_{2,5}$	$\alpha_{2,3}$,$\alpha_{2,4}$,$\alpha_{2,5}$,$\alpha_{2,6}$	$\alpha_{2,4}$	$\alpha_{4,4}$	$\alpha_{2,5}$,$\alpha_{2,6}$
A_5	$2A_1 \oplus A_2$	$3A_1$	$3A_2$	$A_1 \oplus 2A_2$	$A_1 \oplus 2A_2$	$A_2 \oplus A_3$

19	19	21	21	21	22	22 $*$
$H_{6,3}$	$H_{6,3}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,2}$
$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{4,3}$	$\alpha_{2,3}$	$\alpha_{1,1}$,$\alpha_{3,1}$	$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{1,8}$,$\alpha_{2,8}$	$\alpha_{1,6}$,$\alpha_{2,6}$,$\alpha_{1,8}$,$\alpha_{2,8}$,$\alpha_{1,10}$,$\alpha_{2,10}$
$3A_1$	$3A_1$	$3A_1$	$3A_1$	$A_1 \oplus A_3$	$\{0\}$	$2A_2$

23
$H_{6,1}$
α_1,α_5,α_6,α_{24}
$3A_1$

n=6, degeneration $2\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21	22	23 $*$
H	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$	$H_{6,1}$	$H_{6,2}$	$H_{6,3}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{3,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{2,1}$	$\alpha_{1,4}$	$\alpha_{2,1}$	$\alpha_{1,2}$	$\alpha_{1,6}$,$\alpha_{2,6}$	α_{16}
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$A_1 \oplus A_2$	$A_1 \oplus A_2$	$4A_2$	$2A_1 \oplus 2A_2$	D_4	A_3	A_2	$4A_1$

n=6, degeneration $3\mathbb{A}_1$:

j	12	12	12	12	12	12	18	18	18
H	$H_{6,1}$	$H_{6,1}$	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$	$H_{6,1}$	$H_{6,1}$	$H_{6,2}$
orbit of	$\alpha_{2,2}$	$\alpha_{4,2}$	$\alpha_{1,2}$,$\alpha_{6,2}$	$\alpha_{2,2}$	$\alpha_{3,2}$,$\alpha_{5,2}$	$\alpha_{4,2}$	$\alpha_{3,2}$	$\alpha_{1,5}$	$\alpha_{1,2}$,$\alpha_{5,2}$
$(S_{N_j}^{\perp})^{(2)}$	E_6	E_6	D_4	D_4	D_4	D_4	$A_2 \oplus A_5$	$A_1 \oplus A_5$	$3A_1 \oplus A_2$

18	18	18	19	19	19	19
$H_{6,2}$	$H_{6,2}$	$H_{6,2}$	$H_{6,1}$	$H_{6,2}$	$H_{6,2}$	$H_{6,2}$
$\alpha_{2,2}$,$\alpha_{4,2}$	$\alpha_{3,2}$	$\alpha_{1,5}$	$\alpha_{1,3}$,$\alpha_{1,4}$,$\alpha_{1,5}$,$\alpha_{1,6}$	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{1,5}$,$\alpha_{1,6}$
$3A_1 \oplus A_2$	$3A_1 \oplus A_2$	$4A_1$	$A_1 \oplus 3A_2$	$2A_2 \oplus A_3$	$2A_2 \oplus A_3$	$A_1 \oplus A_2 \oplus A_3$

19	19	21	21	21	22
$H_{6,3}$	$H_{6,3}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$
$\alpha_{1,4}$,$\alpha_{3,4}$,$\alpha_{4,4}$	$\alpha_{2,4}$	$\alpha_{2,3}$	$\alpha_{1,5}$,$\alpha_{3,5}$	$\alpha_{2,5}$	$\alpha_{1,1}$,$\alpha_{2,1}$, $\alpha_{1,2}$,$\alpha_{2,2}$,$\alpha_{1,7}$,$\alpha_{2,7}$
D_4	D_4	$2A_1 \oplus A_3$	$2A_1 \oplus A_3$	$2A_1 \oplus A_3$	A_2

22 $*$	23
$H_{6,2}$	$H_{6,1}$
$\alpha_{1,1}$,$\alpha_{2,1}$	α_3,α_7,α_{10},α_{11}
$3A_2$	$4A_1$

n=6, degeneration $6\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21	22
H	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,1}$	$H_{6,2}$	$H_{6,3}$	$H_{6,1}$	$H_{6,2}$
orbit of	$\alpha_{1,2}$	$\alpha_{3,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{1,1}$	$\alpha_{1,1}$	$\alpha_{1,1}$	$\alpha_{1,3}$	$\alpha_{1,2}$,$\alpha_{2,2}$
$(S_{N_j}^{\perp})^{(2)}$	E_6	E_6	$A_2 \oplus A_5$	$A_2 \oplus A_5$	$4A_2$	$2A_2 \oplus A_3$	D_4	$2A_1 \oplus A_3$	$3A_2$

$23*$
$H_{6,1}$
α_2
$4A_1$

$\mathbf{n}$=9, degeneration $2\mathbb{A}_1$:

j	21	21	21	23
H	$H_{9,1}$	$H_{9,2}$	$H_{9,2}$	$H_{9,1}$
orbit of	$\alpha_{1,1}$,$\alpha_{1,2}$,$\alpha_{1,4}$,$\alpha_{1,5}$	$\alpha_{1,1}$,$\alpha_{1,5}$	$\alpha_{2,2}$,$\alpha_{2,3}$,$\alpha_{2,7}$	α_5,α_6
$(S_{N_j}^{\perp})^{(2)}$	$6A_1$	$2A_1$	$4A_1$	$4A_1$

23	$23*$
$H_{9,3}$	$H_{9,4}$
α_6,α_8,α_{10},α_{11},α_{12},α_{15},α_{16}	α_1,α_6,α_8,α_{10},α_{11},α_{12},α_{15},α_{16}
$2A_1$	$\{0\}$

$\mathbf{n}$=9, degeneration $4\mathbb{A}_1$:

j	21	21	23	$23*$
H	$H_{9,1}$	$H_{9,2}$	$H_{9,1}$	$H_{9,4}$
orbit of	$\alpha_{2,3}$	$\alpha_{1,2}$,$\alpha_{1,3}$,$\alpha_{1,7}$	α_2,α_3,α_4,α_7	α_2,α_4
$(S_{N_j}^{\perp})^{(2)}$	$8A_1$	$4A_1$	$4A_1$	$\{0\}$

$\mathbf{n}$=9, degeneration $8\mathbb{A}_1$:

j	21	23	$23*$
H	$H_{9,1}$	$H_{9,2}$	$H_{9,3}$
orbit of	$\alpha_{1,3}$	α_2,α_4	α_2
$(S_{N_j}^{\perp})^{(2)}$	$8A_1$	$8A_1$	$2A_1$

$\mathbf{n}$=10, degeneration $\mathbb{A}_1$:

j	18	18	19	19	21	21	21	21
H	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,2}$	$H_{10,2}$
orbit of	$\alpha_{2,5}$	$\alpha_{4,5}$	$\alpha_{1,3}$,$\alpha_{3,4}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{3,1}$	$\alpha_{2,1}$	$\alpha_{2,1}$	$\alpha_{2,2}$
$(S_{N_j}^{\perp})^{(2)}$	A_1	A_1	$A_1\oplus A_3$	$A_1\oplus A_3$	$3A_1$	$3A_1$	$3A_1$	$3A_1$

22 $*$	23	23
$H_{10,1}$	$H_{10,1}$	$H_{10,2}$
$\alpha_{1,1}$,$\alpha_{2,1}$,$\alpha_{1,2}$,$\alpha_{2,2}$	α_1,α_5,α_6,α_{24}	α_1,α_6
A_2	$3A_1$	A_1

n=10, degeneration $(2\mathbb{A}_1)_I$:

j	18	19 $*$	21	21	21	21	23	23
H	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,2}$	$H_{10,2}$	$H_{10,1}$	$H_{10,2}$
orbit of	$\alpha_{3,1}$,$\alpha_{3,3}$	$\alpha_{3,3}$,$\alpha_{1,4}$	$\alpha_{1,2}$	$\alpha_{2,4}$	$\alpha_{1,1}$	$\alpha_{2,4}$	α_{10},α_{15}	α_{10},α_{16}
$(S_{N_j}^{\perp})^{(2)}$	A_3	$2A_1 \oplus A_3$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	$2A_1$	$4A_1$	$4A_1$	$2A_1$

n=10, degeneration $(2\mathbb{A}_1)_{II}$:

j	18	21	23 $*$
H	$H_{10,1}$	$H_{10,2}$	$H_{10,2}$
orbit of	$\alpha_{1,5}$	$\alpha_{1,2}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$2A_1$	$2A_1$
$\natural\{x \in S_{N_j}^{\perp} \| x^2 = -4\}$	80	80	76

n=10, degeneration $4\mathbb{A}_1$:

j	18	18	19	19	21	21	21	21
H	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$	$H_{10,2}$	$H_{10,2}$
orbit of	$\alpha_{1,1}$,$\alpha_{1,3}$	$\alpha_{2,1}$,$\alpha_{2,3}$	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{1,4}$	$\alpha_{2,3}$	$\alpha_{1,3}$,$\alpha_{3,3}$	$\alpha_{2,3}$
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$2A_3$	$2A_3$	$2A_1 \oplus A_3$	$2A_1 \oplus A_3$	$4A_1$	$4A_1$

21	22	23	23 $*$
$H_{10,2}$	$H_{10,1}$	$H_{10,1}$	$H_{10,2}$
$\alpha_{1,4}$	$\alpha_{1,3}$,$\alpha_{2,3}$,$\alpha_{1,5}$,$\alpha_{2,5}$	α_3,α_7	α_3,α_4,α_5,α_7
$4A_1$	$2A_2$	$4A_1$	$2A_1$

n=10, degeneration $8\mathbb{A}_1$:

j	19	21	23 $*$
H	$H_{10,1}$	$H_{10,1}$	$H_{10,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,3}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_3$	$2A_1 \oplus A_3$	$4A_1$

n=10, degeneration $2\mathbb{A}_2$:

j	$22*$
H	$H_{10,1}$
orbit of	$\alpha_{1,9}$
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$

n=12, degeneration $8\mathbb{A}_1$:

j	18	18	22	$23*$
H	$H_{12,1}$	$H_{12,1}$	$H_{12,1}$	$H_{12,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_1,α_4
$(S_{N_j}^{\perp})^{(2)}$	D_4	D_4	$3A_1 \oplus A_2$	$4A_1$

n=12, degeneration $\mathbb{A}_2$:

j	$22*$
H	$H_{12,1}$
orbit of	$\alpha_{1,5}$,$\alpha_{1,8}$,$\alpha_{1,11}$
$(S_{N_j}^{\perp})^{(2)}$	$2A_1 \oplus A_2$

n=16, degeneration $\mathbb{A}_1$:

j	19	19	20	20	22	23
H	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$
orbit of	$\alpha_{1,3}$,$\alpha_{3,3}$,$\alpha_{4,3}$	$\alpha_{2,3}$	$\alpha_{1,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$,$\alpha_{3,1}$	$\alpha_{1,1}$,$\alpha_{2,1}$,$\alpha_{1,2}$,$\alpha_{2,2}$	α_1,α_6,α_{19},α_{22}
$(S_{N_j}^{\perp})^{(2)}$	$3A_1$	$3A_1$	A_2	A_2	A_2	$3A_1$

n=16, degeneration $5\mathbb{A}_1$:

j	19	19	20	20	22	$23*$
H	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$	$H_{16,1}$
orbit of	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{1,2}$,$\alpha_{4,2}$	$\alpha_{2,2}$,$\alpha_{3,2}$	$\alpha_{1,3}$,$\alpha_{2,3}$,$\alpha_{1,5}$,$\alpha_{2,5}$	α_2,α_3,α_4,α_5
$(S_{N_j}^{\perp})^{(2)}$	D_4	D_4	A_4	A_4	$2A_2$	$4A_1$

n=17, degeneration $\mathbb{A}_1$:

j	19	19	19	19	21	21	21
H	$H_{17,1}$	$H_{17,2}$	$H_{17,2}$	$H_{17,2}$	$H_{17,1}$	$H_{17,1}$	$H_{17,2}$
orbit of	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{2,3}$	$\alpha_{1,4}$	$\alpha_{2,4}$	$\alpha_{1,1}$, $\alpha_{3,1}$,$\alpha_{1,5}$,$\alpha_{3,5}$	$\alpha_{2,1}$,$\alpha_{2,5}$	$\alpha_{2,1}$
$(S_{N_j}^{\perp})^{(2)}$	A_2	D_4	$3A_1 \oplus A_2$	$3A_1 \oplus A_2$	$A_1 \oplus A_3$	$A_1 \oplus A_3$	A_1

21	22	23	23	23 $*$	23
$H_{17,2}$	$H_{17,1}$	$H_{17,1}$	$H_{17,1}$	$H_{17,2}$	$H_{17,3}$
$\alpha_{1,1}$,$\alpha_{3,1}$	$\alpha_{1,1}$,$\alpha_{2,1}$,$\alpha_{1,2}$, $\alpha_{2,2}$	α_5,α_{17},α_{21},α_{24}	α_{15}	α_5,α_6,α_{10},α_{17}	α_{10},α_{15}
A_1	A_2	$4A_1$	$4A_1$	$3A_1$	A_1

n=17, degeneration $3\mathbb{A}_1$:

j	19	19	21	23 $*$	23
H	$H_{17,1}$	$H_{17,2}$	$H_{17,2}$	$H_{17,1}$	$H_{17,3}$
orbit of	$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{1,3}$	$\alpha_{2,2}$	α_1	α_1,α_6
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_2$	$A_1 \oplus D_4$	A_3	$5A_1$	$2A_1$

n=17, degeneration $4\mathbb{A}_1$:

j	19	19	19	21	21	22	23 $*$	23
H	$H_{17,1}$	$H_{17,1}$	$H_{17,2}$	$H_{17,2}$	$H_{17,2}$	$H_{17,1}$	$H_{17,1}$	$H_{17,2}$
orbit of	$\alpha_{1,1}$,$\alpha_{3,1}$,$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{2,1}$	$\alpha_{1,3}$,$\alpha_{3,3}$	$\alpha_{2,3}$	$\alpha_{1,3}$,$\alpha_{2,3}$	α_2	α_2,α_4
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$	$2A_2$	$A_2 \oplus D_4$	A_3	A_3	$2A_2$	$5A_1$	$4A_1$

23
$H_{23,3}$
α_2,α_3,α_8,α_{11}
$2A_1$

n=17, degeneration $6\mathbb{A}_1$:

j	21	21	22	23 $*$
H	$H_{17,1}$	$H_{17,2}$	$H_{17,1}$	$H_{17,2}$
orbit of	$\alpha_{2,2}$	$\alpha_{1,2}$	$\alpha_{1,5}$,$\alpha_{2,5}$	α_1,α_7
$(S_{N_j}^{\perp})^{(2)}$	$2A_3$	A_3	$2A_2$	$4A_1$

n=17, degeneration $12\mathbb{A}_1$:

j	19	21	23 $*$
H	$H_{17,2}$	$H_{17,1}$	$H_{17,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,2}$	α_3
$(S_{N_j}^{\perp})^{(2)}$	$A_2 \oplus D_4$	$2A_3$	$5A_1$

n=18, degeneration $\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21	22 $*$	23
H	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,2}$	$H_{18,2}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$
orbit of	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{3,1}$	$\alpha_{2,5}$	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{2,3}$	$\alpha_{4,3}$	$\alpha_{2,1}$,$\alpha_{2,2}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_1,α_6
$(S_{N_j}^{\perp})^{(2)}$	$3A_1$	$3A_1$	$2A_1 \oplus A_2$	$3A_1$	A_2	A_1	A_1	$3A_1$	$\{0\}$	A_1

n=18, degeneration $2\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21	22 $*$	23
H	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,2}$	$H_{18,2}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{3,1}$	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{2,1}$,$\alpha_{2,5}$	$\alpha_{1,3}$	$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{1,2}$	$\alpha_{1,7}$,$\alpha_{2,7}$	α_{15},α_{16}
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$A_1 \oplus A_2$	$A_1 \oplus A_2$	$2A_2$	$2A_1$	A_3	$2A_1$	A_2	$2A_1$

n=18, degeneration $3\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21
H	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,2}$	$H_{18,2}$	$H_{18,1}$
orbit of	$\alpha_{2,2}$	$\alpha_{4,2}$	$\alpha_{3,2}$	$\alpha_{1,5}$	$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{2,1}$	$\alpha_{3,1}$	$\alpha_{2,3}$,$\alpha_{2,5}$
$(S_{N_j}^{\perp})^{(2)}$	D_4	D_4	$3A_1 \oplus A_2$	$4A_1$	$A_1 \oplus A_2$	A_3	A_3	$4A_1$

22	$23*$
$H_{18,1}$	$H_{18,1}$
$\alpha_{1,2}$,$\alpha_{2,2}$	α_4,α_5
A_2	$2A_1$

n=18, degeneration $6\mathbb{A}_1$:

j	12	12	18	18	19	19	19	21
H	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,1}$	$H_{18,2}$	$H_{18,2}$	$H_{18,1}$
orbit of	$\alpha_{1,2}$	$\alpha_{3,2}$	$\alpha_{1,2}$	$\alpha_{2,2}$	$\alpha_{1,1}$,$\alpha_{1,5}$	$\alpha_{1,1}$	$\alpha_{1,4}$	$\alpha_{1,3}$,$\alpha_{1,5}$
$(S_{N_j}^{\perp})^{(2)}$	D_4	D_4	$3A_1 \oplus A_2$	$3A_1 \oplus A_2$	$2A_2$	A_3	A_3	$4A_1$

22	$23*$
$H_{18,1}$	$H_{18,1}$
$\alpha_{1,5}$,$\alpha_{2,5}$	α_2,α_8
A_2	$2A_1$

n=21, degeneration $4\mathbb{A}_1$:

j	$23*$
H	$H_{21,1}$
orbit of	α_1,α_2,α_3,α_8,α_{12}
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$

n=21, degeneration $16\mathbb{A}_1$:

j	$23*$
H	$H_{21,2}$
orbit of	α_1
$(S_{N_j}^{\perp})^{(2)}$	$8A_1$

n=22, degeneration $2\mathbb{A}_1$:

j	21	23	$23*$
H	$H_{22,1}$	$H_{22,2}$	$H_{22,3}$
orbit of	$\alpha_{1,1}$,$\alpha_{1,2}$	α_{12},α_{16}	α_8,α_{11},α_{12},α_{16}
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$2A_1$	$\{0\}$

n=22, degeneration $4\mathbb{A}_1$:

j	21	21	$23*$	23
H	$H_{22,1}$	$H_{22,1}$	$H_{22,2}$	$H_{22,3}$
orbit of	$\alpha_{2,3}$	$\alpha_{1,4}$	α_6,α_8	α_1,α_2,α_4,α_6
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$	$4A_1$	$2A_1$	$\{0\}$

n=22, degeneration $8\mathbb{A}_1$:

j	21	23	$23*$
H	$H_{22,1}$	$H_{22,1}$	$H_{22,2}$
orbit of	$\alpha_{1,3}$	α_2,α_4	α_2
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$	$4A_1$	$2A_1$

n=26, degeneration $8\mathbb{A}_1$:

j	18	18	22	$23*$
H	$H_{26,1}$	$H_{26,1}$	$H_{26,1}$	$H_{26,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{1,5}$,$\alpha_{2,5}$	α_1,α_3
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$A_1 \oplus A_2$	$2A_1$

n=26, degeneration $2\mathbb{A}_2$:

j	$22*$
H	$H_{26,1}$
orbit of	$\alpha_{1,3}$
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_2$

n=30, degeneration $3\mathbb{A}_1$:

j	19	19	22	$23*$
H	$H_{30,1}$	$H_{30,1}$	$H_{30,2}$	$H_{30,1}$
orbit of	$\alpha_{2,1}$	$\alpha_{1,4}$,$\alpha_{1,5}$,$\alpha_{1,6}$	$\alpha_{1,1}$, $\alpha_{2,1}$,$\alpha_{1,2}$,$\alpha_{2,2}$,$\alpha_{1,6}$,$\alpha_{2,6}$,$\alpha_{1,7}$,$\alpha_{2,7}$	α_4,α_5,α_{11},α_{15}
$(S_{N_j}^{\perp})^{(2)}$	$3A_2$	$A_1 \oplus 2A_2$	$\{0\}$	$3A_1$

n=30, degeneration $9\mathbb{A}_1$:

j	19	22	23 $*$
H	$H_{30,1}$	$H_{30,1}$	$H_{30,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_1
$(S_{N_j}^{\perp})^{(2)}$	$3A_2$	$3A_2$	$3A_1$

n=32, degeneration $2\mathbb{A}_1$:

j	19	20 $*$	23
H	$H_{32,1}$	$H_{32,2}$	$H_{32,1}$
orbit of	$\alpha_{3,3}$	$\alpha_{1,1}$	α_{19}
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$\{0\}$	$2A_1$

n=32, degeneration $5\mathbb{A}_1$:

j	19	19	20	20	22 $*$	23
H	$H_{32,1}$	$H_{32,1}$	$H_{32,2}$	$H_{32,2}$	$H_{32,1}$	$H_{32,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{2,1}$	$\alpha_{1,2}$,$\alpha_{4,2}$	$\alpha_{2,2}$,$\alpha_{3,2}$	$\alpha_{1,3}$,$\alpha_{2,3}$	α_3,α_4
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$2A_1$	$2A_1$	$A_1 \oplus A_2$	$2A_1$

n=32, degeneration $10\mathbb{A}_1$:

j	19	20	23 $*$
H	$H_{32,1}$	$H_{32,1}$	$H_{32,1}$
orbit of	$\alpha_{3,1}$	$\alpha_{1,2}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_4	$2A_1$

n=32, degeneration $5\mathbb{A}_2$:

j	20	22 $*$
H	$H_{32,1}$	$H_{32,1}$
orbit of	$\alpha_{2,2}$	$\alpha_{1,6}$
$(S_{N_j}^{\perp})^{(2)}$	A_4	$A_1 \oplus A_2$

n=33, degeneration $7\mathbb{A}_1$:

j	21	21	23 $*$
H	$H_{33,1}$	$H_{33,1}$	$H_{33,1}$
orbit of	$\alpha_{1,2}$, $\alpha_{3,2}$	$\alpha_{2,2}$	α_1,α_3,α_{11}
$(S_{N_j}^{\perp})^{(2)}$	A_3	A_3	$3A_1$

n=34, degeneration $\mathbb{A}_1$:

j	19	19	19	19	21 $*$	21	21	21
H	$H_{34,1}$	$H_{34,2}$	$H_{34,2}$	$H_{34,2}$	$H_{34,1}$	$H_{34,1}$	$H_{34,1}$	$H_{34,2}$
orbit of	$\alpha_{2,3}$,$\alpha_{2,4}$	$\alpha_{2,3}$	$\alpha_{1,4}$	$\alpha_{2,4}$	$\alpha_{2,1}$	$\alpha_{1,2}$,$\alpha_{3,2}$	$\alpha_{2,2}$	$\alpha_{2,1}$
$(S_{N_j}^{\perp})^{(2)}$	A_2	A_3	$A_1 \oplus A_2$	$A_1 \oplus A_2$	$A_1 \oplus A_3$	$3A_1$	$3A_1$	A_1

21	21	23	23	23	23	23
$H_{34,3}$	$H_{34,3}$	$H_{34,1}$	$H_{34,2}$	$H_{34,2}$	$H_{34,3}$	$H_{34,4}$
$\alpha_{1,2}$,$\alpha_{3,2}$	$\alpha_{2,2}$	α_5,α_6,α_{17},α_{21}	α_6	α_{21},α_{24}	α_5,α_{21}	α_{10},α_{24}
A_1	A_1	$3A_1$	$2A_1$	$2A_1$	A_1	A_1

n=34, degeneration $2\mathbb{A}_1$:

j	19	21	21	23 $*$	23
H	$H_{34,2}$	$H_{34,1}$	$H_{34,2}$	$H_{34,2}$	$H_{34,3}$
orbit of	$\alpha_{3,4}$	$\alpha_{1,1}$	$\alpha_{1,1}$	α_4	α_4
$(S_{N_j}^{\perp})^{(2)}$	$2A_1 \oplus A_2$	A_3	$\{0\}$	$3A_1$	$2A_1$

n=34, degeneration $3\mathbb{A}_1$:

j	19	19	21	21	23 $*$	23
H	$H_{34,1}$	$H_{34,2}$	$H_{34,2}$	$H_{34,3}$	$H_{34,2}$	$H_{34,4}$
orbit of	$\alpha_{1,3}$,$\alpha_{1,4}$	$\alpha_{1,3}$	$\alpha_{2,2}$	$\alpha_{2,1}$	α_2	α_2,α_3
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_2$	$A_1 \oplus A_3$	$2A_1$	A_3	$3A_1$	$2A_1$

n=34, degeneration $4\mathbb{A}_1$:

j	19	19	19	21	21	21	22	23 $*$	23
H	$H_{34,1}$	$H_{34,1}$	$H_{34,2}$	$H_{34,2}$	$H_{34,2}$	$H_{34,3}$	$H_{34,1}$	$H_{34,2}$	$H_{34,3}$
orbit of	$\alpha_{2,1}$	$\alpha_{4,1}$	$\alpha_{2,1}$	$\alpha_{1,4}$,$\alpha_{3,4}$	$\alpha_{2,4}$	$\alpha_{2,4}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_1	α_1,α_{10}
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$	$2A_2$	$A_2 \oplus A_3$	$2A_1$	$2A_1$	A_3	$2A_2$	$3A_1$	$2A_1$

23
$H_{34,4}$
α_1,α_5
$2A_1$

n=34, degeneration $(6\mathbb{A}_1)_I$:

j	21	21	23 $*$
H	$H_{34,1}$	$H_{34,3}$	$H_{34,1}$
orbit of	$\alpha_{2,3}$	$\alpha_{1,1}$	α_1,α_8
$(S_{N_j}^{\perp})^{(2)}$	$2A_1 \oplus A_3$	A_3	$4A_1$

n=34, degeneration $(6\mathbb{A}_1)_{II}$:

j	21	23 $*$
H	$H_{34,2}$	$H_{34,3}$
orbit of	$\alpha_{1,2}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$2A_1$
$\natural\{x \in S_{N_j}^{\perp} \vert x^2 = -4\}$	50	48

n=34, degeneration $8\mathbb{A}_1$:

j	19	21	23 $*$	23
H	$H_{34,1}$	$H_{34,3}$	$H_{34,1}$	$H_{34,4}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,4}$	α_2	α_4
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$	A_3	$4A_1$	$2A_1$

n=34, degeneration $12\mathbb{A}_1$:

j	19	21	23 $*$
H	$H_{34,2}$	$H_{34,1}$	$H_{34,2}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,3}$	α_3
$(S_{N_j}^{\perp})^{(2)}$	$A_2 \oplus A_3$	$2A_1 \oplus A_3$	$3A_1$

n=34, degeneration $6\mathbb{A}_2$:

j	22 $*$
H	$H_{34,1}$
orbit of	$\alpha_{1,2}$
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$

n=39, degeneration $4\mathbb{A}_1$:

j	$23*$
H	$H_{39,2}$
orbit of	α_2,α_3,α_4
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$

n=39, degeneration $8\mathbb{A}_1$:

j	23	$23*$
H	$H_{39,2}$	$H_{39,3}$
orbit of	α_5	α_3,α_5
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$4A_1$

n=39, degeneration $16\mathbb{A}_1$:

j	$23*$
H	$H_{39,1}$
orbit of	α_2
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$

n=40, degeneration $8\mathbb{A}_1$:

j	$23*$
H	$H_{40,1}$
orbit of	α_1,α_3
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$

n=46, degeneration $6\mathbb{A}_1$:

j	$23*$
H	$H_{46,1}$
orbit of	α_1,α_3
$(S_{N_j}^{\perp})^{(2)}$	$3A_1$

n=46, degeneration $9\mathbb{A}_1$:

j	22	$23*$
H	$H_{46,1}$	$H_{46,1}$
orbit of	$\alpha_{1,1}$,$\alpha_{2,1}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_1 \oplus A_2$	$3A_1$

n=46, degeneration $9\mathbb{A}_2$:

j	$22*$
H	$H_{46,2}$
orbit of	$\alpha_{1,1}$
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus 2A_2$

n=48, degeneration $3\mathbb{A}_1$:

j	19	19	$22*$	23
H	$H_{48,1}$	$H_{48,1}$	$H_{48,2}$	$H_{48,1}$
orbit of	$\alpha_{1,3}$	$\alpha_{2,4}$	$\alpha_{1,6}$,$\alpha_{2,6}$,$\alpha_{1,7}$,$\alpha_{2,7}$	α_5,α_7
$(S_{N_j}^{\perp})^{(2)}$	A_1	A_2	$\{0\}$	A_1

n=48, degeneration $6\mathbb{A}_1$:

j	19	22	$23*$
H	$H_{48,1}$	$H_{48,2}$	$H_{48,1}$
orbit of	$\alpha_{1,1}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	A_2	$\{0\}$	A_1

n=48, degeneration $9\mathbb{A}_1$:

j	19	22	$23*$
H	$H_{48,1}$	$H_{48,1}$	$H_{48,1}$
orbit of	$\alpha_{1,4}$	$\alpha_{1,1}$,$\alpha_{2,1}$	α_4
$(S_{N_j}^{\perp})^{(2)}$	A_2	A_2	A_1

n=49, degeneration $4\mathbb{A}_1$:

j	$23*$	23
H	$H_{49,1}$	$H_{49,3}$
orbit of	α_2,α_4	α_2,α_4,α_8,α_{10},α_{11}
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$	A_1

n=49, degeneration $12\mathbb{A}_1$:

j	$23*$
H	$H_{49,1}$
orbit of	α_1
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$

n=49, degeneration $16\mathbb{A}_1$:

j	$23*$
H	$H_{49,2}$
orbit of	α_2
$(S_{N_j}^{\perp})^{(2)}$	$5A_1$

n=51, degeneration $2\mathbb{A}_1$:

j	21	21	$23*$	23	23
H	$H_{51,1}$	$H_{51,2}$	$H_{51,2}$	$H_{51,3}$	$H_{51,4}$
orbit of	$\alpha_{1,1}$, $\alpha_{1,2}$	$\alpha_{1,1}$	α_2,α_{10}	α_2	α_3,α_6
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$\{0\}$	A_1	$2A_1$	$\{0\}$

n=51, degeneration $4\mathbb{A}_1$:

j	21	$23*$	23
H	$H_{51,2}$	$H_{51,2}$	$H_{51,4}$
orbit of	$\alpha_{2,4}$	α_1	α_1,α_5
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	A_1	$\{0\}$

n=51, degeneration $6\mathbb{A}_1$:

j	21	23	$23*$
H	$H_{51,2}$	$H_{51,3}$	$H_{51,4}$
orbit of	$\alpha_{1,2}$	α_4	α_2,α_7
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$2A_1$	$\{0\}$

n=51, degeneration $8\mathbb{A}_1$:

j	21	23	$23*$
H	$H_{51,2}$	$H_{51,1}$	$H_{51,3}$
orbit of	$\alpha_{1,4}$	α_1,α_4	α_1
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	$2A_1$	$2A_1$
$\natural\{x \in S_{N_j}^{\perp} \mid x^2 = -4\}$	22	22	16

n=51, degeneration $12\mathbb{A}_1$:

j	21	$23*$
H	$H_{51,1}$	$H_{51,2}$
orbit of	$\alpha_{1,3}$	α_4
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$	A_1

n=55, degeneration $\mathbb{A}_1$:

j	19	19	22	$22*$	23	23	23	23
H	$H_{55,1}$	$H_{55,1}$	$H_{55,1}$	$H_{55,2}$	$H_{55,1}$	$H_{55,1}$	$H_{55,2}$	$H_{55,2}$
orbit of	$\alpha_{1,4},\ \alpha_{3,4},\ \alpha_{4,4}$	$\alpha_{2,4}$	$\alpha_{1,10},\ \alpha_{2,10}$	$\alpha_{1,6},\ \alpha_{2,6},\ \alpha_{1,8},\ \alpha_{2,8}$	$\alpha_1,\ \alpha_{10}$	α_{24}	$\alpha_1,\ \alpha_{21},\ \alpha_{24}$	α_5
$(S_{N_j}^{\perp})^{(2)}$	$3A_1$	$3A_1$	$\{0\}$	A_2	$2A_1$	$2A_1$	$3A_1$	$3A_1$

n=55, degeneration $5\mathbb{A}_1$:

j	19	22	23	$23*$
H	$H_{55,1}$	$H_{55,1}$	$H_{55,1}$	$H_{55,2}$
orbit of	$\alpha_{2,1}$	$\alpha_{1,6},\ \alpha_{2,6}$	α_2	α_4
$(S_{N_j}^{\perp})^{(2)}$	D_4	A_2	$3A_1$	$4A_1$

n=55, degeneration $6\mathbb{A}_1$:

j	22	$23*$
H	$H_{55,1}$	$H_{55,1}$
orbit of	$\alpha_{1,1}\ \alpha_{2,1}$	α_4
$(S_{N_j}^{\perp})^{(2)}$	A_2	$3A_1$

n=55, degeneration $10\mathbb{A}_1$:

j	22	23 $*$
H	$H_{55,2}$	$H_{55,1}$
orbit of	$\alpha_{1,1}$, $\alpha_{2,1}$	α_3
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$	$3A_1$

n=55, degeneration $15\mathbb{A}_1$:

j	19	23 $*$
H	$H_{55,1}$	$H_{55,2}$
orbit of	$\alpha_{1,1}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	D_4	$4A_1$

n=56, degeneration $8\mathbb{A}_1$:

j	23 $*$
H	$H_{56,1}$
orbit of	α_1, α_6
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$

n=56, degeneration $16\mathbb{A}_1$:

j	23 $*$
H	$H_{56,2}$
orbit of	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$

n=61, degeneration $3\mathbb{A}_1$:

j	19	23 $*$
H	$H_{61,1}$	$H_{61,1}$
orbit of	$\alpha_{1,5}$, $\alpha_{1,6}$	α_7, α_{16}
$(S_{N_j}^{\perp})^{(2)}$	$A_1 \oplus A_2$	$2A_1$

n=61, degeneration $12\mathbb{A}_1$:

j	19	23 $*$
H	$H_{61,1}$	$H_{61,1}$
orbit of	$\alpha_{1,1}$	α_2
$(S_{N_j}^{\perp})^{(2)}$	$2A_2$	$2A_1$

n=65, degeneration $4\mathbb{A}_1$:

j	23 $*$	23
H	$H_{65,3}$	$H_{65,4}$
orbit of	α_3, α_{10}	α_3, α_4, α_7
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$	A_1

n=65, degeneration $8\mathbb{A}_1$:

j	23 $*$	23
H	$H_{65,2}$	$H_{65,4}$
orbit of	α_1	α_1
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$	A_1

n=65, degeneration $12\mathbb{A}_1$:

j	23 $*$
H	$H_{65,3}$
orbit of	α_1
$(S_{N_j}^{\perp})^{(2)}$	$2A_1$

$\mathbf{n}$=65, degeneration $16\mathbb{A}_1$:

j	$23*$
H	$H_{65,1}$
orbit of	α_1
$(S_{N_j}^{\perp})^{(2)}$	$3A_1$

$\mathbf{n}$=75, degeneration $16\mathbb{A}_1$:

j	$23*$
H	$H_{75,1}$
orbit of	α_1
$(S_{N_j}^{\perp})^{(2)}$	$4A_1$

5 Transcendental lattices $T = S_{L_{K3}}^{\perp}$.

For the fixed type $\mathbf{n}$ of a finite symplectic automorphism group G and the fixed type of degeneration of codimension one P (Dynkin diagram), a general Käherian K3 surface X has the Picard lattice $S_X \cong S$ described in Theorem 1. Its transcendental lattice T_X is the orthogonal complement $T_X = (S_X)_{H^2(X,\mathbb{Z})}^{\perp}$ where $H^2(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$. It is unique up to isomorphisms, and we denote its isomorphism class as L_{K3}.

Thus $T_X \cong T = (S)_{L_{K3}}^{\perp}$ where $S \subset L_{K3}$ is some primitive embedding. By epmimorphicity of Torelli map for K3 surfaces, any such primitive embedding corresponds to K3 surfaces. By Proposition 2, the transcendental lattice T can be any lattice with invariants $(3, 19 - \operatorname{rk} S, q_T \cong -q_S)$ which are equivalent to the genus of T.

We use the following theorem from [14] which follows from results by M. Kneser.

Theorem 3. *([14, Theorem 1.13.2]) An even lattice K with invariants $(t_{(+)}, t_{(-)}, q)$ is unique if simulataneously*

1) $t_{(+)} \geq 1$, $t_{(-)} \geq 1$, $t_{(+)} + t_{(-)} \geq 3$;

2) for each $p \neq 2$, either $\operatorname{rk} K \geq l(A_{q_p}) + 2$, or

$$q_p \cong q_{\theta_1}^{(p)}(p^k) \oplus q_{\theta_2}^{(p)}(p^k) \oplus q_p';$$

3) for $p = 2$, either $\operatorname{rk} K \geq l(A_{q_2}) + 2$, or $q_2 \cong u_+^{(2)}(2^k) \oplus q_2'$, or $q_2 \cong v_+^{(2)}(2^k) \oplus q_2'$, or

$$q_2 \cong q_{\theta_1}^{(2)}(2^k) \oplus q_{\theta_2}^{(2)}(2^{k+1}) \oplus q_2'.$$

From this Theorem, we then obtain

Theorem 4. *For a fixed type* $\mathbf{n}$ *of a finite symplectic automorphism group* G *and the fixed type of degeneration of codimension one* P *(Dynkin diagram), a general Käherian K3 surface* X *has a unique, up to isomorphisms, transcendental lattice* $T_X \cong T = (S)^{\perp}_{L_{K3}}$ *if* $\operatorname{rk} S \le 18$ *(by Theorem 3).*

If $\operatorname{rk} S = 19$ *(equivalently,* $\operatorname{rk} T = 3$*, and then* T *is positive definite), then the isomorphism class of the transcendental lattice* T *is given in the Table 3 below. The transcendental lattice* T *is unique except* $(\mathbf{n} = 55, 10\mathbb{A}_1)$ *when there are two possible isomorphism classes.*

Thus, for $(\mathbf{n} = 55, 10\mathbb{A}_1)$ *(equivlaently,* $G \cong \mathfrak{A}_5$ *and the degeneration has the type* $10\mathbb{A}_1$*) there are two non-equivalent degenerations of codimension one of Kählerian K3 surfaces which have non-isomorphic transcendental lattices.*

Proof. The genus of S and then T is calculated in Table 1 above.

If $\operatorname{rk} T \ge 4$, it satisfies Theorem 3.

If $\operatorname{rk} T = 3$, we calculate T in Table 3 using known tables of positive definite lattices of the rank 2 and 3 for small determinant (determinants ≤ 50 are enough). See [3, Ch. 15, Sects 3, 10]. $\square$

Table 3: Transcendental lattices of the rank 3 of degenerations of codimension 1 of Kählerian K3 surfaces with finite symplectic automorphism groups $G = Clos(G)$.

n	$\|G\|$	i	G	Deg	q_T	T
26	16	8	SD_{16}	$8\mathbb{A}_1$	$2^{+1}_{-1}, 4^{+1}_{-1}, 16^{-1}_{-3}$	$\langle 2\rangle \oplus \langle 4\rangle \oplus \langle 16\rangle$
				$2\mathbb{A}_2$	$2^{-1}_{-5}, 8^{-2}_{II}$	$\left\langle \begin{array}{ccc} 6 & 2 & -2 \\ 2 & 6 & 2 \\ -2 & 2 & 6 \end{array} \right\rangle$
32	20	3	$Hol(C_5)$	$2\mathbb{A}_1$	$4^{+1}_{-1}, 5^{+3}$	$A_3(-5)$
				$5\mathbb{A}_1$	$2^{+3}_{-1}, 5^{-2}$	$\left\langle \begin{array}{ccc} 10 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{array} \right\rangle$
				$10\mathbb{A}_1$	$4^{-1}_{3}, 5^{+2}$	$\left\langle \begin{array}{ccc} 4 & 2 & 2 \\ 2 & 6 & 1 \\ 2 & 1 & 6 \end{array} \right\rangle$
				$5\mathbb{A}_2$	$2^{-1}_{3}, 5^{-2}$	$\left\langle \begin{array}{ccc} 4 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 4 \end{array} \right\rangle$
33	21	1	$C_7 \rtimes C_3$	$7\mathbb{A}_1$	$2^{+1}_{-1}, 7^{+2}$	$\left\langle \begin{array}{ccc} 14 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 2 \end{array} \right\rangle$
46	36	9	3^2C_4	$6\mathbb{A}_1$	$4^{+1}_{1}, 3^{-1}, 9^{+1}$	$\langle 36\rangle \oplus A_2(-1)$
				$9\mathbb{A}_1$	$2^{-3}_{-5}, 3^{+2}$	$\langle 2\rangle \oplus \langle 6\rangle \oplus \langle 6\rangle$
				$9\mathbb{A}_2$	$2^{-1}_{-5}, 3^{+2}$	$\langle 6\rangle \oplus A_2(-1)$
48	36	10	$\mathfrak{S}_{3,3}$	$3\mathbb{A}_1$	$2^{+3}_{3}, 3^{-2}, 9^{+1}$	$\langle 6\rangle \oplus A_2(-6)$
				$6\mathbb{A}_1$	$4^{+1}_{-1}, 3^{+2}, 9^{+1}$	$\left\langle \begin{array}{ccc} 6 & 0 & 3 \\ 0 & 6 & 3 \\ 3 & 3 & 12 \end{array} \right\rangle$
				$9\mathbb{A}_1$	$2^{-3}_{1}, 3^{-3}$	$3\,\langle 6\rangle$
51	48	48	$C_2 \times \mathfrak{S}_4$	$2\mathbb{A}_1$	$4^{+3}_{-1}, 3^{+2}$	$\langle 4\rangle \oplus \langle 12\rangle \oplus \langle 12\rangle$
				$4\mathbb{A}_1$	$2^{+2}_{II}, 8^{+1}_{-1}, 3^{+2}$	$\left\langle \begin{array}{ccc} 8 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 8 \end{array} \right\rangle$
				$6\mathbb{A}_1$	$4^{-3}_{1}, 3^{-1}$	$\langle 4\rangle \oplus A_2(-4)$
				$8\mathbb{A}_1$	$2^{-2}_{II}, 4^{-1}_{3}, 3^{+2}$	$\langle 12\rangle \oplus A_2(-2)$
				$12\mathbb{A}_1$	$2^{-2}_{II}, 8^{+1}_{1}, 3^{1}$	$\langle 8\rangle \oplus A_2(-2)$

$\mathbf{n}$	$\lvert G\rvert$	i	G	Deg	q_T	T
55	60	5	$\mathfrak{A}_5$	$\mathbb{A}_1$	$2_1^{-3}, 3^{-1}, 5^{-2}$	$\langle 2\rangle \oplus A_2(-10)$
				$5\mathbb{A}_1$	$2_5^{+3}, 3^{-1}, 5^{+1}$	$\left\langle \begin{array}{ccc} 6 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 6 \end{array} \right\rangle$
				$6\mathbb{A}_1$	$4_{-1}^{+1}, 5^{-2}$	$\left\langle \begin{array}{ccc} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 8 \end{array} \right\rangle$
				$10\mathbb{A}_1$	$4_1^{+1}, 3^{-1}, 5^{-1}$	$\left\langle \begin{array}{ccc} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 16 \end{array} \right\rangle, \left\langle \begin{array}{ccc} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{array} \right\rangle$
				$15\mathbb{A}_1$	$2_3^{+3}, 5^{-1}$	$\langle 2\rangle \oplus \langle 2\rangle \oplus \langle 10\rangle$
56	64	138	$\Gamma_{25}a_1$	$8\mathbb{A}_1$	$4_4^{-2}, 8_3^{-1}$	$\langle 4\rangle \oplus \langle 4\rangle \oplus \langle 8\rangle$
				$16\mathbb{A}_1$	4_3^{+3}	$3\,\langle 4\rangle$
61	72	43	$\mathfrak{A}_{4,3}$	$3\mathbb{A}_1$	$2_3^{-1}, 4_{II}^{+2}, 3^{+2}$	$\langle 6\rangle \oplus A_2(-4)$
				$12\mathbb{A}_1$	$8_{-1}^{+1}, 3^{+2}$	$\left\langle \begin{array}{ccc} 2 & 0 & 1 \\ 0 & 6 & 3 \\ 1 & 3 & 8 \end{array} \right\rangle$
65	96	227	2^4D_6	$4\mathbb{A}_1$	$4_{-3}^{-3}, 3^{+1}$	$\langle 4\rangle \oplus \langle 4\rangle \oplus \langle 12\rangle$
				$8\mathbb{A}_1$	$2_{II}^{-2}, 8_1^{+1}, 3^{+1}$	$\langle 8\rangle \oplus A_2(-2)$
				$12\mathbb{A}_1$	4_3^{+3}	$3\,\langle 4\rangle$
				$16\mathbb{A}_1$	$2_{II}^{+2}, 4_5^{-1}, 3^{+1}$	$\langle 4\rangle \oplus A_2(-2)$
75	192	1023	$4^2\mathfrak{A}_4$	$16\mathbb{A}_1$	$2_{II}^{-2}, 8_3^{-1}$	$A_3(-2)$

6 Connected components of moduli of Kählerian K3 surfaces with a negative definite Picard lattice M and a transcendental lattice K

Here we apply our results in [13], [15] about description of connected components of moduli of Kählerian K3 surfaces with conditions on Picard lattice. Using these methods and results, we want to describe connected components of moduli of Kählerian K3 surfaces X such that the Picard lattice S_X contains a fixed negative definite primitive sublattice M and the orthogonal complement $M^{\perp}_{H^2(X,\mathbb{Z})} = K$ where the lattice K is also fixed. A general such K3 surface X has the Picard lattice $S_X = M$ and the transcendental lattice $T_X = K$.

The lattices M and K are orthogonal complements to each other in the even unimodular lattice $H^2(X,\mathbb{Z})$. This defines a canonical isomorphism $\varphi : q_S \cong -q_K$ which is equivalent to the natural finite index extension $M \oplus K \subset H^2(X,\mathbb{Z})$:

$$H^2(X,\mathbb{Z}) = \{m^* \oplus k^* \mid m^* \in M^*, k^* \in K^*,\ \varphi(m^* + M) = k^* + K\}.$$

Periods

$$H^{2,0}(X) + H^{1,1}(X) + H^{0,2}(X)$$

of X are equivalent to the positive definite 2-dimensional oriented subspace

$$\Pi_2(X) = (H^{2,0}(X) + H^{0,2}(X)) \cap K \otimes \mathbb{R} \subset K \otimes \mathbb{R}$$

(its orientation is equivalent to the natural orientation of the 1-dimensional complex space $H^{2,0}(X)$). Kähler class $c(X)$ of X defines a half $V^+(X)$ of the cone

$$V(X) = \{x \in H^{1,1}(X)_{\mathbb{R}} \mid x^2 > 0\}$$

containing $c(X)$. Together, they define continuously changing orientations in all 3-dimensional positive definite subspaces $\Pi_3 \subset H^2(X,\mathbb{Z}) \otimes \mathbb{R}$ and $\Pi_3 \subset K \otimes \mathbb{R}$ such that an oriented basis of $\Pi_2(X)$ together with $c(X)$ define an oriented basis in $\Pi_2(X) + \mathbb{R}c(X)$. We denote this orientation as $o(X) \in \{+,-\}$ and the corresponding $K \otimes \mathbb{R}$ with the choice of such orientation as $(K \otimes \mathbb{R})_{o(X)}$. Moreover, effective elements $\delta \in M$ with $\delta^2 = -2$ define a fundamental decomposition $P(X) : \Delta(M) = \Delta^+(M) \cup -\Delta^+(M)$ of the set $\Delta(M)$ of roots of M with square -2 where elements of $\Delta^+(M)$ are effective.

We can consider the 4-tuple

$$(P(X), \varphi(X), o(X), \Pi_2(X))$$

as periods of a marked K3 surface X with $M \subset S_X$ and $M^{\perp}_{H^2(X,\mathbb{Z})} = K$.

Let $W^{(2)}(M)$ be the group generated by reflections in elements of $\Delta(M)$. It acts identically on the discriminant group and the discriminant form q_M. By changing $M \subset S_X$ by $w : M \to M \subset S_X$ where $w \in W^{(2)}(M)$, we can assume that $P(X) = P$ where the decomposition $P : \Delta(M) = \Delta^+(M) \cup -\Delta^+(M)$ is fixed. This does not changes $\varphi(X), o(X), \Pi_2(X)$. Let $w_0 \in W^{(2)}(M)$ changes the decomposition $P = \Delta^+(M) \cup -\Delta^+(M)$ to $-P = -\Delta^+(M) \cup \Delta^+(M)$. Then, by changing $M \subset S_X$ by $-w_0 : M \to M \subset S_X$, $K \subset M^\perp_{H^2(X,\mathbb{Z})}$ by $-id_K : K \to K \subset M^\perp_{H^2(X,\mathbb{Z})}$, if necessary, we can assume that $o(X) = +$ is fixed. Here, we use that 3 is odd. Thus, periods of marked in this way K3 surfaces are given by the pair

$$(\varphi(X), \Pi_2(X) \subset (K \otimes \mathbb{R})_+).$$

It follows that for a fixed isomorphism $\varphi(X) = \varphi : q_M \cong -q_K$ the spaces of periods and moduli of such K3 surfaces are connected by Global Torelli Theorem [19], [2] and epimorphicity of period map [7], [22], [21] for K3 surfaces.

By changing markings to $g : M \to M \subset S_X$, $f^{-1} : K \to K \subset M^\perp_{H^2(X,\mathbb{Z})}$ by $g \in O(M)$ with $g(P) = P$ and by $f \in O(K)$, periods will be changed by equivalent periods

$$(\overline{f}\varphi\overline{g}, (f \otimes \mathbb{R})(\Pi_2(X)) \subset (K \otimes \mathbb{R})_{f(+)})$$

and moduli.

Thus, we obtain

Theorem 5. *The number of connected components of moduli of Kählerian K3 surfaces X with Picard lattice M where $M < 0$, and a transcendental lattice isomorphic to K (further we shall call them as* **weak connected components***) is equal to the number of double cosets $\overline{O(K)}\backslash O(q)/\overline{O(M)}$ where $\overline{O(M)}$ and $\overline{O(K)}$ are images of $O(M)$ and $O(K)$ in $O(q)$ where $q = q_M = -q_K$.*

Here we consider primitive embeddings $f_1 : M \subset S_X$ and $f_2 : M \subset S_X$ as equivalent if they are different by an automorphism of the lattice M.

We remark that the double cosets $\overline{O(K)}\backslash O(q)/\overline{O(M)}$ of the Theorem 5 are equivalent to isomorphism classes of primitive embeddings of the lattice M into L_{K3} with $M^\perp_{L_{K3}} \cong K$ where two such primitive embeddings $f_1 : M \subset L_{K3}$, $f_2 : M \subset L_{K3}$ are equivalent if $f_2(M) = h(f_1(M))$ for $h \in O(L_{K3})$. See Proposition 2. Such isomorphism classes are preserved under continuous deformations of K3 surfaces since they are discrete data of the deformations.

Similarly, we obtain

Theorem 6. *The number of connected components of moduli of Kählerian K3 surfaces X with Picard lattice M where $M < 0$, and fixed $P(X) = P = \Delta^+(M) \cup -\Delta^+(M)$, and a*

transcendental lattice isomorphic to K (further we shall call them as **strong connected components***) is equal to the number of left cosets $\overline{O^+(K)}\backslash O(q_K)$ (equivalently, to the index $[O(q_K) : \overline{O^+(K)}]$) where $O^+(K) \subset O(K)$ consists of automorphisms which preserve orientations $(K \otimes \mathbb{R})_+$ and $(K \otimes \mathbb{R})_-$.*

Here we consider primitive embeddings $f_1 : M \subset S_X$ and $f_2 : M \subset S_X$ as equivalent if they are equal.

We remark that the left cosets $\overline{O^+(K)}\backslash O(q_K)$ of Theorem 6 are equivalent to all isomorphism classes of primitive embeddings of the lattice M into L_{K3} with $M^{\perp}_{L_{K3}} \cong K$, and choices of orientation $(M^{\perp}_{L_{K3}} \otimes \mathbb{R})_\alpha$, $\alpha = \pm$ where two such data $f_1 : M \subset L_{K3}$, α_1 and $f_2 : M \subset L_{K3}$, α_2 are equivalent if $f_2 = hf_1$ and $h(\alpha_1) = \alpha_2$ for some $h \in O(L_{K3})$.

In R. Miranda and D.R. Morrison [8], [9] (announcement) and D.G. James [6] (proofs), for indefinite lattices K of Theorem 6 (equivalently, if $\mathrm{rk}\, K \geq 4$), the sum

$$e_{--}(K) = \sum_{K' \in g(K)} [O(q_{K'}) : \overline{O^+(K')}] \tag{8}$$

is calculated in terms of invariants of the genus $g(K)$ of K (as a particular case of general results which generalize our [14, Theorem 1.14.2]). In [8], [9], the group $O^+(K)$ is denoted as $O_{--}(K)$. See Theorem in [9, page 31].

7 Connected components of moduli of degenerations of codimension one of Kählerian K3 surfaces with finite symplectic automorphism groups

Using results of Sec. 6, we obtain

Theorem 7. *For a fixed type* **n** *of a finite symplectic automorphism group G and a fixed type of degeneration of codimension one P (Dynkin diagram), a general Käherian K3 surface X has a unique (with few exceptions), up to isomorphisms, Picard lattice $S_X \cong S$ described in Theorem 1 and a transcendental lattice $T_X \cong T = (S)^{\perp}_{L_{K3}}$ for some primitive embedding $S \subset L_{K3}$ described in Theorem 4.*

The number M_w of weak connected components of moduli of general Kähelerian K3 surfaces X for such S and T is equal to the number of double cosets

$$M_w = \natural(\overline{O(T)}\backslash O(q)/\overline{O(S)}) \tag{9}$$

where $q_S = q = -q_T$, in notations of Theorem 5.

Theorem 8. *For a fixed type* $\mathbf{n}$ *of a finite symplectic automorphism group* G *and a fixed type of degeneration of codimension one* P *(Dynkin diagram), a general Käherian K3 surface* X *has a unique (with few exceptions), up to isomorphisms, Picard lattice* $S_X \cong S$ *described in Theorem 1 and a transcendental lattice* $T_X \cong T = (S)^{\perp}_{L_{K3}}$ *for some primitive embedding* $S \subset L_{K3}$ *described in Theorem 4.*

(1) Assume that $\operatorname{rk} S \le 18$ *(equivalently,* $\operatorname{rk} T \ge 4$ *and then* T *is indefinite). Then* T *is unique, up to isomorphisms, and the number* M_s *of strong connected components of moduli of general Kähelerian K3 surfaces* X *for such* S *is equal to*

$$M_s = [O(q_T) : \overline{O^+(T)}].$$

For all these cases, $M_s = 1$ *(and then* $M_w = 1$*) except cases*

$$(\mathbf{n}, P) = (12, \mathbb{A}_2),\ (16, \mathbb{A}_1),\ (18, 2\mathbb{A}_1),\ (22, 2\mathbb{A}_1),\ (34, 2\mathbb{A}_1),\ (39, 4\mathbb{A}_1),\ (40, 8\mathbb{A}_1), \qquad (10)$$

when $M_s = 2$*. Moreover, for all cases* (10)*, we have* $[\underline{O(q_T)} : \overline{O(T)}] = 1$ *and then* $M_w = 1$ *(by Theorem 7), except* $(\mathbf{n}, P) = (16, \mathbb{A}_1)$ *when* $[O(q_T) : \overline{O(T)}] = 2$*.*

(2) Assume that $\operatorname{rk} S = 19$ *(equivalently,* $\operatorname{rk} T = 3$ *and then* T *is positive definite) and* $(\mathbf{n}, P) \neq (55, 10\mathbb{A}_1)$*. Then* T *is unique, up to isomorphisms, and*

$$M_s = [O(q_T) : \overline{O^+(T)}] = |O(q_T)|/|O^+(T)/W^+(T)|$$

where $+$ *means "proper" (with determinant=1) automorphisms of* T*, and* $W(T) \subset O(T)$ *is generated by reflections in all roots with square* 2 *of* T*. If* $(\mathbf{n}, P) = (55, 10\mathbb{A}_1)$*, then*

$$M_s = |O(q_{T_1})|/|O^+(T_1)/W^+(T_1)| + |O(q_{T_2})|/|O^+(T_2)/W^+(T_2)|$$

where T_1 *and* T_2 *are two non-isomorphic transcendental lattices of this case.*

Exact calculations of these invariants are given in Table 4 below.

Proof. In case (1) when T is indefinite and of rank $\operatorname{rk} T \ge 4$, results follow from Theorem in [9, page 31]. For example, let us consider the case $(\mathbf{n}, P)) = (18, 2\mathbb{A}_1)$. By Table 1, $\operatorname{rk} S = 17$ and $q_S \cong 2^{+2}_{II}, 4^{+1}_{7}, 3^{+4}$. Then $\operatorname{rk} T = 22 - 17 = 5$ and $q_T = -q_S \cong 2^{+2}_{II}, 4^{+1}_{1}, 3^{+4}$. In notations of Theorem in [9, page 31], for $p = 2$, we have $s(0) = \operatorname{rk} T - l(q_{T_2}) = 2 > 0$; for $p = 3$, we have $\operatorname{rk} T - l(q_{T_3}) = 5 - 4 = 1$, $3 \bmod 4 = 3$, $\det(T) = 2^4 \cdot 3^4$, $\det(K(q_{T_3})) \equiv 3^4 \bmod (\mathbb{Z}_3^*)^2$ and $\Delta = \det(T)/\det(K(q_{T_3})) \equiv 2^4 \bmod (\mathbb{Z}_3^*)^2$, $\left(\frac{2\Delta}{3}\right) = -1$. By Theorem in [9, page 31], then $e_2 = 1$, $f_2 = 4$, $e_3 = 2$, $f_3 = 2$, type$= (-,-)$, $e_{++}(T) = e_2 \cdot e_3 = 1 \cdot 2 = 2$, the group $\widetilde{\Sigma}(T) = \{(+,+),(-,-)\}$, $e_{--}(T) = [O(q_T) : \underline{\overline{O^+(T)}}] = e_{++}(T) = 2$ (the group $O^+(T)$ is denoted as $O_{--}(T)$ in [8], [9]), $e(T) = [O(q_T) : \overline{O(T)}] = (1/2)e_{++}(T) = 1$. Thus, $M_s = 2$ and $M_w = 1$ in this case.

Let us consider the case (2) when T is positive definite and $\operatorname{rk} T = 3$. By [14, Remark 1.14.6], the kernel of the natural homomorphism $\pi : O(T) \to O(q_T)$ is equal to $W(T)$ because $\operatorname{rk} T = 3 < 8$. It follows that the kernel of the natural homomorphism $\pi : O^+(T) \to O(q_T)$ is equal to $W^+(T)$. Then the order of $|\overline{O^+(T)}| = |\pi(O^+(T))| = |O^+(T)/W^+(T)|$. It follows the statement. $\square$

We hope to present calculations of missing numbers M_w of weak connected components of moduli in further variants of the paper and further publications.

Table 4: Automorphism groups of transcendental lattices of rank 3 of degenerations of codimension 1 of Kählerian K3 surfaces with finite symplectic automorphism groups $G = Clos(G)$ and strong connected components of moduli.

n	$\|G\|$	i	G	Deg	$\|O(q_T)\|$	$\|O(T)\|$	$\|W(T)\|$	M_s
26	16	8	SD_{16}	$8\mathbb{A}_1$	16	8	2	4
				$2\mathbb{A}_2$	96	48	1	4
32	20	3	$Hol(C_5)$	$2\mathbb{A}_1$	480	48	1	20
				$5\mathbb{A}_1$	24	8	1	6
				$10\mathbb{A}_1$	16	16	1	2
				$5\mathbb{A}_2$	12	12	1	2
33	21	1	$C_7 \rtimes C_3$	$7\mathbb{A}_1$	16	8	2	4
46	36	9	3^2C_4	$6\mathbb{A}_1$	24	24	6	6
				$9\mathbb{A}_1$	16	8	2	4
				$9\mathbb{A}_2$	8	24	6	2
48	36	10	$\mathfrak{S}_{3,3}$	$3\mathbb{A}_1$	432	24	1	36
				$6\mathbb{A}_1$	288	16	1	36
				$9\mathbb{A}_1$	288	48	1	12
51	48	48	$C_2 \times \mathfrak{S}_4$	$2\mathbb{A}_1$	256	16	1	32
				$4\mathbb{A}_1$	64	16	1	8
				$6\mathbb{A}_1$	192	24	1	16
				$8\mathbb{A}_1$	96	24	1	8
				$12\mathbb{A}_1$	96	24	1	8
55	60	5	$\mathfrak{A}_5$	$\mathbb{A}_1$	144	24	2	12
				$5\mathbb{A}_1$	24	8	1	6
				$6\mathbb{A}_1$	24	8	2	6
				$10\mathbb{A}_1$	8	16, 8	4, 1	2, 2
				$15\mathbb{A}_1$	12	16	4	3
56	64	138	$\Gamma_{25}a_1$	$8\mathbb{A}_1$	64	16	1	8
				$16\mathbb{A}_1$	96	48	1	4
61	72	43	$\mathfrak{A}_{4,3}$	$3\mathbb{A}_1$	96	24	1	8
				$12\mathbb{A}_1$	16	8	2	4

n	$\lvert G\rvert$	i	G	Deg	$\lvert O(q_T)\rvert$	$\lvert O(T)\rvert$	$\lvert W(T)\rvert$	M_s
65	96	227	2^4D_6	$4\mathbb{A}_1$	64	16	1	8
				$8\mathbb{A}_1$	96	24	1	8
				$12\mathbb{A}_1$	96	48	1	4
				$16\mathbb{A}_1$	24	24	1	2
75	192	1023	$4^2\mathfrak{A}_4$	$16\mathbb{A}_1$	48	48	1	2

8 Appendix: Programs

Here we give Programs 7 and 8 for GP/PARI Calculator, Version 2.7.0 which were used for calculations above. They also include Programs 1 - 6 from [16] — [18].

Program 7: niemeier\genwithorbit.txt

```
\\for a Niemeier lattice N_i given by root matrix r
\\and cord matrix cord, R=r^-1
\\and subgroup H\subset A(N_i)
\\and its orbits ORB matrix, each line gives
\\orbit of length > 1
\\it calculates all additional
\\1-elements orbits to matrix ORBF and prints it
\\(1-elements orbits the last)
\\it calculates coinvariant sublattice N_H
\\together with morb-orbit given by its first
\\element morb of the orbit as SUBLpr below
\\by its rational generators,
\\and checks if
\\it has primitive embedding to L_K3
\\Then it prints invariants of its discriminant group DSUBLpr below
\\and the intersection matrix rSUBLpr of SUBLpr
sORB=matsize(ORB);m1=0;
for(k1=1,sORB[1],for(k2=1,sORB[2],\
if(ORB[k1,k2]==0,,m1=m1+1)));
ORBF=matrix(sORB[1]+(24-m1),sORB[2]);
for(k=1,sORB[1],ORBF[k,]=ORB[k,]);
l=sORB[1];
for(t=1,24,mu=0;for(k1=1,sORB[1],for(k2=1,sORB[2],\
if(ORB[k1,k2]!=t,,mu=1)));if(mu==1,,l=l+1;ORBF[l,1]=t));\
```

```
print("ORBF=",ORBF);\
SUBL0=matrix(24,24);alpha=0;\
for(k1=1,sORB[1],for(k2=1,sORB[2]-1,\
if(ORB[k1,k2+1]>0,alpha=alpha+1;\
SUBL0[,alpha]=R[,ORB[k1,k2]]-R[,ORB[k1,k2+1]])));\
SUBL0[morb,24]=1;\
sORBF=matsize(ORBF);\
SUBL=SUBL0;\
a=matrix(24,24+matsize(cord)[1]);\
for(i=1,24,a[i,i]=1);for(i=1,matsize(cord)[1],a[,24+i]=cord[i,]~);\
L=a;N=SUBL;\
ggg=gcd(N);N1=N/ggg;\
M=L;\
gg=gcd(M);M1=M/gg;\
ww=matsnf(M1,1);uu=ww[1];vv=ww[2];dd=ww[3];\
mm=matsize(dd)[1];nn=matsize(dd)[2];\
nnn=nn;for(i=1,nn,if(dd[,i]==0,nnn=nnn-1));\
VV=matrix(nn,nnn);\
nnnn=0;for(i=1,nn,if(dd[,i]==0,,nnnn=nnnn+1;VV[,nnnn]=vv[,i]));\
M2=M1*VV;MM=M2*gg;\
kill(gg);kill(M1);kill(ww);kill(uu);kill(vv);kill(dd);kill(mm);\
kill(nn);kill(nnn);kill(nnnn);kill(M2);\
L1=MM;kill(VV);\
N2=L1^-1*N1;\
ww=matsnf(N2,1);uu=ww[1];vv=ww[2];dd=ww[3];\
N3=N2*vv;mm=matsize(dd)[1];nn=matsize(dd)[2];\
nnn=nn;for(i=1,nn,if(dd[,i]==0,nnn=nnn-1));\
N4=matrix(mm,nnn);\
nnnn=0;\
for(i=1,nn,if(dd[,i]==0,,nnnn=nnnn+1;\
ddd=gcd(dd[,i]);N4[,nnnn]=N3[,i]/ddd));\
Npr=L1*N4;\
kill(ggg);kill(N1);kill(M);kill(L1);kill(MM);\
kill(N2);kill(ww);kill(uu);kill(vv);kill(dd);\
kill(N3);kill(mm);kill(nn);kill(nnn);kill(nnnn);\
kill(ddd);kill(N4);\
SUBLpr1=Npr;\
R=r;B=SUBLpr1;\
```

```
l=B~*R*B;\
ww=matsnf(l,1);uu=ww[1];vv=ww[2];dd=ww[3];\
nn=matsize(l)[1];nnn=nn;for(i=1,nn,if(dd[i,i]==0,nnn=nnn-1));\
b=matrix(nn,nnn,X,Y,vv[X,Y+nn-nnn]);\
ll=b~*l*b;\
d=vector(nnn,X,dd[X+nn-nnn,X+nn-nnn]);\
kill(ww);kill(uu);kill(vv);kill(dd);\
kill(nn);kill(nnn);\
BB=B*b;G=BB~*R*BB;D=d;\
SUBLpr=BB;DSUBLpr=D;rSUBLpr=G;
```

Program 8: niemeier\genus6.txt

```
\\for a non-degenerate lattice
\\L given by a symmetric integer matrix l
\\in some generators
\\calculates the elementary difisors (Smyth) basis of L
\\as a matrix b and
\\calculates the matrix ll=b~*l*b
\\of L in the bases b
\\calculates invariants d of L\subset L^\ast
\\for primes, p, calculates llll=L\otimes \bz_p
\\in Smith form
\\thus, calculates genus of L
ww=matsnf(l,1);uu=ww[1];vv=ww[2];dd=ww[3];
nn=matsize(l)[1];nnn=nn;for(i=1,nn,if(dd[i,i]==0,nnn=nnn-1));
b=matrix(nn,nnn,X,Y,vv[X,Y+nn-nnn]);
ll=b~*l*b;
d=vector(nnn,X,dd[X+nn-nnn,X+nn-nnn]);
kill(ww);kill(uu);kill(vv);kill(dd);
kill(nn);kill(nnn);
n=matsize(d)[2];
delta=vector(n,X,d[n+1-X]);
bb=matrix(n,n,X,Y,b[X,n+1-Y]);
lll=bb~*l*bb;
F=factor(delta[n]);
f1=matsize(F)[1];
for(KK1=1,f1,p=F[KK1,1];t=F[KK1,2];\
u=vector(n);\
for(k2=1,n,u[k2]=gcd(delta[k2],p^t));\
v=vector(n);j1=1;\
v[j1]=1;for(k2=2,n,if(u[k2]>u[k2-1],j1=j1+1;v[j1]=k2,));\
vv=vector(j1,X,v[X]);\
nvv=matsize(vv)[2];\
for(k4=1,nvv,if(k4<nvv,ss=vv[k4+1]-vv[k4];\
cc=matrix(ss,ss,X,Y,lll[vv[k4]+X-1,vv[k4]+Y-1]/u[vv[k4]]);\
ccdet=matdet(cc);dcc=Mod(ccdet,p);kron=kronecker(ccdet,p);\
if(p>2,print(u[v[k4]]," size=",ss," det=",dcc," kro=",kron),),\
ss=n-v[k4]+1;cc=matrix(ss,ss,X,Y,lll[vv[k4]+X-1,vv[k4]+Y-1]/u[vv[k4]]);\
```

```
ccdet=matdet(cc);dcc=Mod(ccdet,p);kron=kronecker(ccdet,p);\
if(p>2,print(u[v[k4]]," size=",ss," det=",dcc," kro=",kron),)));\
if(p!=2,,llll=lll;for(k1=1,nvv,\
if(k1<nvv,ss1=vv[k1+1]-vv[k1],ss1=n-vv[k1]+1);\
cc1=matrix(ss1,ss1,X,Y,llll[vv[k1]+X-1,vv[k1]+Y-1]);\
ty=0;for(k=1,ss1,if(Mod(cc1[k,k]/u[vv[k1]],2)==Mod(0,2),,ty=1));\
dcc1=Mod(matdet(cc1/u[vv[k1]]),8);\
a=cc1/u[vv[k1]];\
na=matsize(a)[1];\
alpha==1;\
while(alpha,\
t=0;beta=0;mu=1;\
for(k=1,na,if(Mod(a[k,k],2)==Mod(1,2)&&mu=1,t=k,mu=0));\
for(k=1,t,for(k1=k+1,na,if(a[k,k1]==0,,beta=1)));\
for(k=t+1,na,if(Mod(a[k,k],2)==Mod(0,2),,beta=1));\
if(beta==1,,alpha=0);\
t1=0;gam1=1;for(k=1,na,if(Mod(a[k,k],2)==Mod(1,2)&&gam1==1,\
for(k1=k+1,na,if(a[k,k1]==0&&gam1==1,,gam1=0)),gam1=0);\
if(gam1==1,t1=t1+1,));\
t2=t1;gam2=1;\
for(k=t1+1,na,if(Mod(a[k,k],2)==Mod(1,2)&&gam2==1,t2=k;gam2=0,));\
if(t2==t1,,\
if(t2==t1+1,,trans=matrix(na,na);for(k=1,na,trans[k,k]=1);\
trans[t2,t1+1]=1;trans[t1+1,t1+1]=0;trans[t1+1,t2]=1;trans[t2,t2]=0;\
a=trans~*a*trans);\
trans=matrix(na,na);for(k=1,na,trans[k,k]=1);\
for(m=t1+2,na,trans[t1+1,m]=-a[t1+1,m]/a[t1+1,t1+1]);\
a=trans~*a*trans));\
sign8=Mod(0,8);\
for(k=1,t,sign8=sign8+Mod(a[k,k],8));\
print(u[vv[k1]]," size=",ss1," type=",ty," det=",dcc1," sign8=",sign8);\
kill(alpha);kill(sign8);kill(t);kill(t1);kill(t2);kill(a);kill(trans);\
kill(beta);kill(mu);kill(gam1);kill(gam2);kill(na);\
ccc1=cc1^-1;\
for(k2=k1+1,nvv,if(k2<nvv,ss2=vv[k2+1]-vv[k2],ss2=n-vv[k2]+1);\
cc21=matrix(ss1,ss2,X,Y,llll[vv[k1]+X-1,vv[k2]+Y-1]);\
cc21n=ccc1*cc21;ttt=matrix(n,n);for(aaa1=1,n,ttt[aaa1,aaa1]=1);\
for(aa1=1,ss1,for(aa2=1,ss2,ttt[vv[k1]+aa1-1,vv[k2]+aa2-1]=-cc21n[aa1,aa2]));\
```

llll=ttt~*llll*ttt))));

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